The Bending-Gradient theory for laminates and in-plane periodic plates

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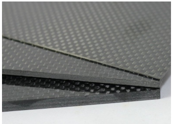
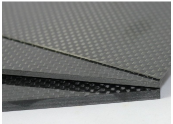
The open question of shear forces in heterogeneous plates



- Thick or Thin plates?
  - Thin Plate – Kirchhoff-Love – $\varphi_\alpha = U_{3,\alpha}$:
    asymptotic derivation, transverse shear effects neglected
    (Kirchhoff (1850); Love (1888); Ciarlet and Destuynder (1979))
  - Thick Plate – Reissner-Mindlin – $\varphi_\alpha \neq U_{3,\alpha}$:
    axiomatic and controversial. Natural boundary conditions!
    (Reissner (1944); Hencky (1947); Mindlin (1951))

. . . *plates are generalized continua!*

The open question of shear forces in heterogeneous plates



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    axiomatic and controversial. Natural boundary conditions!
    (Reissner (1944); Hencky (1947); Mindlin (1951))
- Deriving formally Reissner-Mindlin plate model from asymptotic expansions?

$\Rightarrow$ The Bending-Gradient plate model

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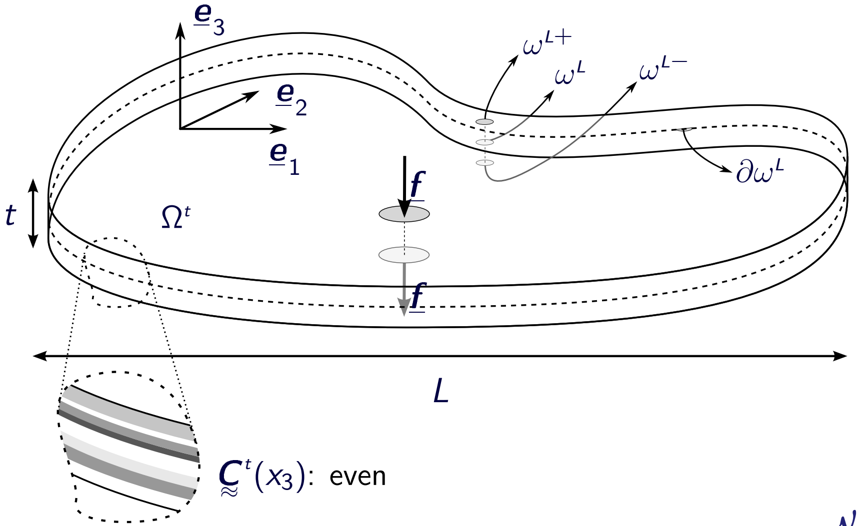
- The 3D Problem
- The asymptotic expansions for a laminated plate
- The Kirchhoff-Love plate model
- The Bending-Gradient plate model
- Applications of the Bending-Gradient theory to laminates
- Periodic plates

The 3D problem configuration



$\underset{\approx}{C}^t(x_3)$: even

The 3D problem equations

$$\begin{cases} \sigma^t_{ij,j} = 0 \quad \text{on } \Omega^t. \\ \sigma^t_{ij} = C^t_{ijkl}(x_3)\varepsilon^t_{kl} \quad \text{on } \Omega^t. \\ \sigma^t_{i3} = \pm f_i \quad \text{on } \omega^{L\pm}. \\ \varepsilon^t_{ij} = u^t_{(i,j)} \quad \text{on } \Omega^t. \\ u^t_i = 0 \quad \text{on } \partial\omega^L \times]-t/2, t/2[\end{cases}$$

$$\begin{cases} \underset{\sim}{\sigma}^t \cdot \underline{\boldsymbol{\nabla}} = 0 \quad \text{on } \Omega^t. \\ \underset{\sim}{\sigma}^t\left(\underline{\boldsymbol{x}}\right) = \underset{\approx}{\boldsymbol{C}}^t(x_3) : \underset{\sim}{\varepsilon}^t\left(\underline{\boldsymbol{x}}\right) \quad \text{on } \Omega^t. \\ \underset{\sim}{\sigma}^t \cdot \left(\pm\underline{\boldsymbol{e}}_3\right) = \underline{\boldsymbol{f}} \quad \text{on } \omega^{L\pm}. \\ \underset{\sim}{\varepsilon}^t = \underline{\boldsymbol{u}}^t \otimes^s \underline{\boldsymbol{\nabla}} \quad \text{on } \Omega^t. \\ \underline{\boldsymbol{u}}^t = 0 \quad \text{on } \partial\omega^L \times]-t/2, t/2[\end{cases}$$

- monoclinic and even $\underset{\approx}{\boldsymbol{C}}^t$:
  $C^t_{\alpha\beta\gamma3} = C^t_{333\alpha} = 0$,
  $\alpha, \beta, \gamma, ... = 1, 2$.
- symmetrically laminated plate
- symmetric transverse load
  $\underline{\boldsymbol{f}} = f_3 \underline{\boldsymbol{e}}_3$

$\Rightarrow$ pure bending:

- u^t_3 and $\sigma^t_{\alpha3}$ even / x_3
- u^t_α, $\sigma^t_{\alpha\beta}$ and σ^t_{33} odd / x_3

Proof of skew symmetry

Let $\underline{\boldsymbol{u}}^{t'}$ be the image of $\underline{\boldsymbol{u}}^{t}$ by the symmetry with respect to the (x_1, x_2)-plane

$$\underline{\boldsymbol{u}}^{t'}(x_1, x_2, x_3) = \begin{pmatrix} +u_1^t(x_1, x_2, -x_3) \\ +u_2^t(x_1, x_2, -x_3) \\ -u_3^t(x_1, x_2, -x_3) \end{pmatrix}$$

Obviously, $\underline{\boldsymbol{u}}^{t'} = 0 \quad$ on $\partial\omega^L \times]-t/2, t/2[$. Its corresponding strain $\underset{\sim}{\boldsymbol{\varepsilon}}^{t'} = \underline{\boldsymbol{u}}^{t'} \otimes^s \underline{\boldsymbol{\nabla}}$ is:

$$\underset{\sim}{\boldsymbol{\varepsilon}}^{t'}(x_1, x_2, x_3) = \begin{pmatrix} \varepsilon_{11}^t & \varepsilon_{12}^t & -\varepsilon_{13}^t \\ \varepsilon_{12}^t & \varepsilon_{22}^t & -\varepsilon_{23}^t \\ -\varepsilon_{13}^t & -\varepsilon_{23}^t & \varepsilon_{33}^t \end{pmatrix} (x_1, x_2, -x_3)$$

Proof of skew symmetry

Its corresponding stress $\underset{\sim}{\boldsymbol{\sigma}}^{t'}\left(\underline{\boldsymbol{x}}\right) = \underset{\approx}{\boldsymbol{C}}^{t}(x_3) : \underset{\sim}{\boldsymbol{\varepsilon}}^{t'}\left(\underline{\boldsymbol{x}}\right)$ is:

$$\begin{bmatrix} \sigma^{t'}_{11} \\ \sigma^{t'}_{22} \\ \sigma^{t'}_{12} \\ \sigma^{t'}_{13} \\ \sigma^{t'}_{23} \\ \sigma^{t'}_{33} \end{bmatrix} = \begin{bmatrix} C^t_{1111} & C^t_{1122} & C^t_{1112} & 0 & 0 & C^t_{1133} \\ & C^t_{2222} & C^t_{2212} & 0 & 0 & C^t_{2233} \\ & & C^t_{1212} & 0 & 0 & C^t_{1233} \\ & & & C^t_{1313} & C^t_{1323} & 0 \\ & \text{SYM} & & & C^t_{2323} & 0 \\ & & & & & C^t_{3333} \end{bmatrix} \begin{bmatrix} \varepsilon^{t'}_{11} \\ \varepsilon^{t'}_{22} \\ 2\varepsilon^{t'}_{12} \\ 2\varepsilon^{t'}_{13} \\ 2\varepsilon^{t'}_{23} \\ \varepsilon^{t'}_{33} \end{bmatrix}$$

Because $\underset{\approx}{\boldsymbol{C}}^{t}(x_3)$ is even and monoclinic, then:

$$\underset{\sim}{\boldsymbol{\sigma}}^{t'}(x_1, x_2, x_3) = \begin{pmatrix} \sigma^t_{11} & \sigma^t_{12} & -\sigma^t_{13} \\ \sigma^t_{12} & \sigma^t_{22} & -\sigma^t_{23} \\ -\sigma^t_{13} & -\sigma^t_{23} & \sigma^t_{33} \end{pmatrix}(x_1, x_2, -x_3)$$

Proof of skew symmetry

Finally, the balance equation $\underset{\sim}{\sigma}^{t'} \cdot \underline{\boldsymbol{\nabla}} = 0$ is easy to check and we have:

$$\underset{\sim}{\sigma}^{t} \cdot \left(\pm \underline{\boldsymbol{e}}_3 \right) = -\underline{\boldsymbol{f}} = -f_3 \underline{\boldsymbol{e}}_3 \quad \text{on } \omega^{L\pm}.$$

Therefore,

$$\underline{\boldsymbol{u}}^{t'}\left(\underline{\boldsymbol{x}}\right) = -\underline{\boldsymbol{u}}^{t}\left(\underline{\boldsymbol{x}}\right), \; \underset{\sim}{\varepsilon}^{t'}\left(\underline{\boldsymbol{x}}\right) = -\underset{\sim}{\varepsilon}^{t}\left(\underline{\boldsymbol{x}}\right), \; \underset{\sim}{\sigma}^{t'}\left(\underline{\boldsymbol{x}}\right) = -\underset{\sim}{\sigma}^{t}\left(\underline{\boldsymbol{x}}\right)$$

Hence,

$$\begin{pmatrix} +u_1^t(x_1, x_2, -x_3) \\ +u_2^t(x_1, x_2, -x_3) \\ -u_3^t(x_1, x_2, -x_3) \end{pmatrix} = \begin{pmatrix} -u_1^t(x_1, x_2, x_3) \\ -u_2^t(x_1, x_2, x_3) \\ -u_3^t(x_1, x_2, x_3) \end{pmatrix} \ldots$$

Variational formulation

The set of statically compatible stress fields is:

$$SC^{3D,t}: \begin{cases} \underset{\sim}{\sigma}^t \cdot \underline{\boldsymbol{\nabla}} = 0 \text{ on } \Omega^t \\ \underset{\sim}{\sigma}^t \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = \underline{\boldsymbol{f}} \text{ on } \omega^{L\pm}, \end{cases}$$

The set of kinematically compatible displacement fields is:

$$KC^{3D,t}: \begin{cases} \underset{\sim}{\varepsilon}^t = \underline{\boldsymbol{u}}^t \otimes^s \underline{\boldsymbol{\nabla}} \text{ on } \Omega^t \\ \underline{\boldsymbol{u}}^t = 0 \text{ on } \partial\omega^L \times]-t/2, t/2[\end{cases}$$

The strain and stress energy density w^{3D} and w^{*3D} are respectively given by:

$$w^{3D}\left(\underset{\sim}{\varepsilon}\right) = \frac{1}{2}\underset{\sim}{\varepsilon} : \underset{\approx}{\boldsymbol{C}}^t : \underset{\sim}{\varepsilon}, \qquad w^{*3D}\left(\underset{\sim}{\sigma}\right) = \frac{1}{2}\underset{\sim}{\sigma} : \underset{\approx}{\boldsymbol{S}}^t : \underset{\sim}{\sigma}$$

with:

$$\underset{\approx}{\boldsymbol{S}}^t = \left(\underset{\approx}{\boldsymbol{C}}^t\right)^{-1}$$

Potential energy

$$P^{3D}\left(\underset{\approx}{\varepsilon}^t\right) = \min_{\underset{\approx}{\varepsilon} \in KC^{3D,t}} \left\{ P^{3D}\left(\underset{\approx}{\varepsilon}\right) \right\}$$

The potential energy P^{3D} is given by:

$$P^{3D}\left(\underset{\approx}{\varepsilon}\right) = \int_{\Omega^t} w^{3D}\left(\underset{\approx}{\varepsilon}\right) d\Omega^t - \int_{\omega^L} f_3\left(u_3^+ + u_3^-\right) d\omega^L$$

$\underline{\boldsymbol{u}}^{\pm} = \underline{\boldsymbol{u}}(x_1, x_2, \pm t/2)$ are the 3D displacement fields on the upper and lower faces of the plate.

Complementary energy

$$P^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}^{t}\right) = \min_{\underset{\sim}{\boldsymbol{\sigma}} \in SC^{3D,t}} \left\{ P^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}\right) \right\}$$

The complementary potential energy P^{*3D} given by:

$$P^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}\right) = \int_{\Omega^t} w^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}\right) d\Omega^t$$

At the solution (Clapeyron's formula):

$$P^{3D}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^{t}\right) + P^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}^{t}\right) = 0$$

Building a plate model?

For typical width L and thickness t, let $\frac{t}{L} \to 0$

- Solve a 2D problem, called the "plate problem"
- "fair" 3D displacement localization
- "fair" 3D stress localization

Exercice: Trial from plate equilibrium equations...

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Differential system depending on a small parameter

We want to solve the following differential equation on $[0,1]$:

$$u''(x) - \eta u(x) = 0,\ u(0) = 0,\ u(1) = a$$

where $\eta > 0$ is a small parameter.

The solution is trivial:

$$u^\eta(x) = a\frac{\sinh\left(\sqrt{\eta}x\right)}{\sinh\left(\sqrt{\eta}\right)}$$

The limit of $u^\eta(x)$ as η goes to 0^+ is:

$$\lim_{\eta\to 0^+} u^\eta(x) = ax.$$

Taylor's series

Using Taylor's series:

$$
\begin{aligned}
\sinh\left(\sqrt{\eta}x\right) &= \frac{\left(\sqrt{\eta}x\right)^1}{1!} + \frac{\left(\sqrt{\eta}x\right)^3}{3!} + \cdots \\
\sinh\left(\sqrt{\eta}\right) &= \frac{\left(\sqrt{\eta}\right)^1}{1!} + \frac{\left(\sqrt{\eta}\right)^3}{3!} + \cdots
\end{aligned}
$$

We obtain:

$$
u^{\eta}(x) = ax + \ a\eta^1 \frac{x^3 - x}{3!} + a\eta^2 \left(\frac{x^5 - x}{5!} - \frac{x^3 - x}{3!3!}\right) + \cdots
$$

The method

- Write $u^\eta(x)$ as a series:

$$u^\eta(x) = u^0(x) + \eta^1 u^1(x) + \ldots \cdots + \eta^i u^i(x) + \cdots$$

  where u^i are unknown functions.

- Inject this series in the differential system

$$u''(x) - \eta u(x) = 0,\ u(0) = 0,\ u(1) = a$$

  and make null all the terms in η^i.

- Solve the cascade system which determines the u^i

Resolution

The cascade system is:

$$
\begin{array}{llll}
\text{Term in } \eta^0 & : \quad u^{0\prime\prime}(x) = 0, & u^0(0) = 0, & u^0(1) = a \\
\text{Term in } \eta^1 & : \quad u^{1\prime\prime}(x) = u^0, & u^1(0) = 0, & u^1(1) = 0 \\
 & \quad \cdots\cdot & & \\
\text{Term in } \eta^i & : \quad u^{i\prime\prime}(x) = u^{i-1}, & u^i(0) = 0, & u^i(1) = 0
\end{array}
$$

The solution is obtained by mathematical induction:

$$
u^0(x) = ax,\ u^1(x) = a\frac{x^3 - x}{3!},\ u^2(x) = a\frac{x^5 - x}{5!} - a\frac{x^3 - x}{3!3!}, \cdots
$$

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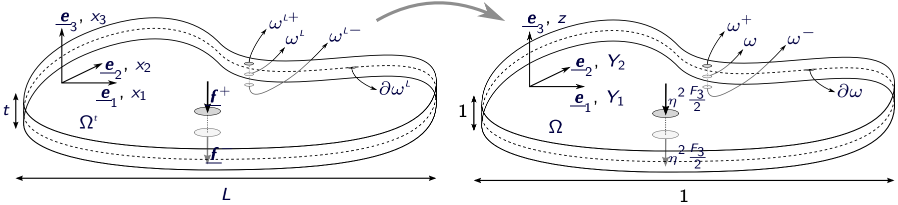
Change of variables



- $Y_\alpha = \dfrac{x_\alpha}{L}$ for the in-plane variables, $Y_\alpha \in \omega$
- $z = \dfrac{x_3}{t}$ for the out-of-plane variable, $z \in]-\frac{1}{2}, \frac{1}{2}[$
- $\eta = \dfrac{t}{L}$ is the small parameter

The fourth-order elasticity tensor can be rewritten as:

$$\underset{\approx}{\boldsymbol{C}}^{\,t}(x_3) = \underset{\approx}{\mathbb{C}}\left(t^{-1}x_3\right) = \underset{\approx}{\mathbb{C}}(z)$$

Non-dimensional fields

We define the non-dimensional fields $(\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})$ as follows:

$$\begin{cases} \underline{\boldsymbol{u}}^t & (x_1, x_2, x_3) = & L\,\underline{\boldsymbol{u}} & (x_1/L, x_2/L, x_3/t) = & L\,\underline{\boldsymbol{u}} & (Y_1, Y_2, z) \\ \underset{\sim}{\varepsilon}^t & (x_1, x_2, x_3) = & \underset{\sim}{\varepsilon} & (x_1/L, x_2/L, x_3/t) = & \underset{\sim}{\varepsilon} & (Y_1, Y_2, z) \\ \underset{\sim}{\boldsymbol{\sigma}}^t & (x_1, x_2, x_3) = & \underset{\sim}{\boldsymbol{\sigma}} & (x_1/L, x_2/L, x_3/t) = & \underset{\sim}{\boldsymbol{\sigma}} & (Y_1, Y_2, z) \end{cases}$$

The derivation rule for these fields is:

$$\begin{aligned} \underline{\boldsymbol{\nabla}} &= \left(\frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}\right) \\ &= L^{-1}\left(\frac{\partial}{\partial Y_1}, \frac{\partial}{\partial Y_2}, 0\right) + t^{-1}\left(0, 0, \frac{\partial}{\partial z}\right) = L^{-1}\underline{\boldsymbol{\nabla}}_Y + t^{-1}\underline{\boldsymbol{\nabla}}_z \\ &= L^{-1}\underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)} \end{aligned}$$

where

$$\underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)} := \underline{\boldsymbol{\nabla}}_Y + \frac{1}{\eta}\,\underline{\boldsymbol{\nabla}}_z$$

Natural scaling of the stress

$$\begin{cases} \sigma^t_{\alpha\beta,\beta} + \sigma^t_{\alpha 3,3} = 0 \\ \sigma^t_{\alpha 3,\alpha} + \sigma^t_{33,3} = 0 \\ \sigma^t_{33}(\pm t/2) = \pm f_3 \\ \sigma^t_{\alpha 3}(\pm t/2) = 0 \end{cases} \Rightarrow \begin{cases} \sigma^t_{\alpha 3} = -\int_{-t/2}^{x_3} \sigma^t_{\alpha\beta,\beta} du \\ \sigma^t_{33} = -\int_{-t/2}^{x_3} \sigma^t_{\alpha 3,\alpha} du - f_3 \end{cases}$$

$$\sigma^t_{\alpha\beta} \sim \eta^0 \quad \Rightarrow \quad \sigma^t_{\alpha 3} \sim \eta^1, \quad \sigma^t_{33} \sim \eta^2 \quad \text{and} \quad f_3 \sim \eta^2$$

The out-of-plane loading is scaled as:

$$\underline{\boldsymbol{f}}(x_1, x_2) = \eta^2 \frac{F_3\left(Y_1, Y_2\right)}{2} \underline{\boldsymbol{e}}_3$$

The non-dimensional 3D problem

The set of statically compatible fields can be rewritten as:

$$SC^{3D}: \begin{cases} \underset{\sim}{\sigma} \cdot \underline{\nabla}^{\eta}_{(Y,z)} = 0 \text{ on } \Omega = \omega \times]-\frac{1}{2}, +\frac{1}{2}[, \\ \underset{\sim}{\sigma} \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = \frac{\eta^2}{2} F_3 \underline{\boldsymbol{e}}_3 \quad \text{on } \omega^{\pm} \end{cases}$$

The kinematically compatible fields becomes:

$$KC^{3D}: \begin{cases} \underset{\sim}{\varepsilon} = \underline{\boldsymbol{u}} \otimes^{s} \underline{\nabla}^{\eta}_{(Y,z)} \text{ on } \Omega, \\ \underline{\boldsymbol{u}} = 0 \quad \text{on } \partial\omega \times]-\frac{1}{2}, +\frac{1}{2}[\end{cases}$$

The constitutive law becomes:

$$\underset{\sim}{\sigma}\left(Y_1, Y_2, z\right) = \underset{\approx}{\mathbb{C}}\left(z\right) : \underset{\sim}{\varepsilon}\left(Y_1, Y_2, z\right)$$

$$\underline{\nabla}^{\eta}_{(Y,z)} = \underline{\nabla}_{Y} + \frac{1}{\eta} \underline{\nabla}_{z}$$

Properties of the non-dimensional solution

For given $\left(\omega, \underset{\approx}{\mathbb{C}}, F_3, \eta\right)$ where $\underset{\approx}{\mathbb{C}}$ is monoclinic and even in z, and under some regularity conditions, the solution of the non-dimensional problem is unique.

Obviously, due the change of variables $x_3 \to z$:

- u_3 and $\sigma_{\alpha 3}$ are even in z
- u_α, $\sigma_{\alpha\beta}$ and σ_{33} are odd in z

We have the following new properties:

- u_3 and $\sigma_{\alpha 3}$ are odd in η
- u_α, $\sigma_{\alpha\beta}$ and σ_{33} are even in η

Proof

New change of variable $z' = -\frac{x_3}{t}$ for the out-of-plane variable. The new non-dimensional fields $(\underline{\boldsymbol{u}}', \underset{\sim}{\varepsilon}', \underset{\sim}{\boldsymbol{\sigma}}')$ are defined by:

$$\begin{cases} \underline{\boldsymbol{u}}^t & (x_1, x_2, x_3) = & L\,\underline{\boldsymbol{u}}' & (x_1/L, x_2/L, -x_3/t) = & L\,\underline{\boldsymbol{u}}' & (Y_1, Y_2, z') \\ \underset{\sim}{\varepsilon}^t & (x_1, x_2, x_3) = & \underset{\sim}{\varepsilon}' & (x_1/L, x_2/L, -x_3/t) = & \underset{\sim}{\varepsilon}' & (Y_1, Y_2, z') \\ \underset{\sim}{\boldsymbol{\sigma}}^t & (x_1, x_2, x_3) = & \underset{\sim}{\boldsymbol{\sigma}}' & (x_1/L, x_2/L, -x_3/t) = & \underset{\sim}{\boldsymbol{\sigma}}' & (Y_1, Y_2, z') \end{cases}$$

The new derivation rule for these fields is:

$$\begin{aligned} \underline{\boldsymbol{\nabla}} &= \left(\frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}\right) \\ &= L^{-1}\left(\frac{\partial}{\partial Y_1}, \frac{\partial}{\partial Y_2}, 0\right) - t^{-1}\left(0, 0, \frac{\partial}{\partial z'}\right) = L^{-1}\underline{\boldsymbol{\nabla}}_Y - t^{-1}\underline{\boldsymbol{\nabla}}_{z'} \\ &= L^{-1}\underline{\boldsymbol{\nabla}}^{-\eta}_{(Y,z')} \end{aligned}$$

where

$$\underline{\boldsymbol{\nabla}}^{-\eta}_{(Y,z')} = \underline{\boldsymbol{\nabla}}_Y - \frac{1}{\eta}\,\underline{\boldsymbol{\nabla}}_{z'}$$

The new equations

$$SC^{3D'}: \begin{cases} \underset{\sim}{\boldsymbol{\sigma}}' \cdot \underline{\boldsymbol{\nabla}}^{-\eta}_{(Y,z')} = 0 \text{ on } \Omega = \omega \times \left]-\frac{1}{2}, +\frac{1}{2}\right[, \\ \underset{\sim}{\boldsymbol{\sigma}}' \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = -\frac{\eta^2}{2} F_3 \underline{\boldsymbol{e}}_3 \quad \text{on } \omega^\pm \end{cases}$$

$$KC^{3D'}: \begin{cases} \underset{\sim}{\boldsymbol{\varepsilon}}' = \underline{\boldsymbol{u}}' \otimes^s \underline{\boldsymbol{\nabla}}^{-\eta}_{(Y,z')} \text{ on } \Omega, \\ \underline{\boldsymbol{u}}' = 0 \quad \text{on } \partial\omega \times \left]-\frac{1}{2}, +\frac{1}{2}\right[\end{cases}$$

$$\underset{\sim}{\boldsymbol{\sigma}}'\left(Y_1, Y_2, z'\right) = \underset{\approx}{\mathbb{C}}\left(z'\right) : \underset{\sim}{\boldsymbol{\varepsilon}}'\left(Y_1, Y_2, z'\right)$$

Effect of the transformation $\eta \to -\eta$ on the non-dimensional solution

The new non-dimensional fields $(\underline{\boldsymbol{u}}', \underset{\sim}{\varepsilon}', \underset{\sim}{\boldsymbol{\sigma}}')$ are solutions of the same equations as for $(\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})$ where $F_3 \to -F_3$ and $\eta \to -\eta$:

$$(\underline{\boldsymbol{u}}', \underset{\sim}{\varepsilon}', \underset{\sim}{\boldsymbol{\sigma}}')\left(Y_1, Y_2, z'\right) = (\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})^{(-F_3,-\eta)}\left(Y_1, Y_2, z'\right)$$

Moreover, by definition, the new non-dimensional fields coincide with the initial ones with $z = -z'$:

$$(\underline{\boldsymbol{u}}', \underset{\sim}{\varepsilon}', \underset{\sim}{\boldsymbol{\sigma}}')\left(Y_1, Y_2, z'\right) = (\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})^{(F_3,\eta)}\left(Y_1, Y_2, -z'\right)$$

Hence, we have:

$$(\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})^{(-\eta)}(Y_1, Y_2, z) = -(\underline{\boldsymbol{u}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\boldsymbol{\sigma}})^{(\eta)}(Y_1, Y_2, -z)$$

Even components in z are odd in η and odd components in z are even in η.

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Expansion

We assume

$$\begin{cases} \underline{\boldsymbol{u}} = \eta^{-1}\underline{\boldsymbol{u}}^{-1} + \eta^{0}\underline{\boldsymbol{u}}^{0} + \eta^{1}\underline{\boldsymbol{u}}^{1} + \cdots \\ \underset{\sim}{\boldsymbol{\varepsilon}} = \eta^{0}\underset{\sim}{\boldsymbol{\varepsilon}}^{0} + \eta^{1}\underset{\sim}{\boldsymbol{\varepsilon}}^{1} + \cdots \\ \underset{\sim}{\boldsymbol{\sigma}} = \eta^{0}\underset{\sim}{\boldsymbol{\sigma}}^{0} + \eta^{1}\underset{\sim}{\boldsymbol{\sigma}}^{1} + \cdots \end{cases}$$

$\underline{\boldsymbol{u}}^p, \underset{\sim}{\boldsymbol{\varepsilon}}^p$ and $\underset{\sim}{\boldsymbol{\sigma}}^p$, $p = -1, 0, 1, 2...$, are functions of (Y_1, Y_2, z)

Because:

- u_3 and $\sigma_{\alpha 3}$ are odd in η
- u_α, $\sigma_{\alpha\beta}$ and σ_{33} are even in η

we have:

- u_3^p and $\sigma_{\alpha 3}^p$ are null for even p and even in z for odd p.
- u_α^p, $\sigma_{\alpha\beta}^p$ and σ_{33}^p are null for odd p and odd in z for even p.

Statics

The normalized 3D equilibrium equation becomes:

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)} = \eta^{-1} \left(\underset{\sim}{\boldsymbol{\sigma}}^{0} \cdot \underline{\boldsymbol{\nabla}}_{z} \right) + \eta^{0} \left(\underset{\sim}{\boldsymbol{\sigma}}^{0} \cdot \underline{\boldsymbol{\nabla}}_{Y} + \underset{\sim}{\boldsymbol{\sigma}}^{1} \cdot \underline{\boldsymbol{\nabla}}_{z} \right) + \cdots = 0.$$

Hence,

$$\underset{\sim}{\boldsymbol{\sigma}}^{0} \cdot \underline{\boldsymbol{\nabla}}_{z} = 0 \quad \text{and} \quad \underset{\sim}{\boldsymbol{\sigma}}^{p} \cdot \underline{\boldsymbol{\nabla}}_{Y} + \underset{\sim}{\boldsymbol{\sigma}}^{p+1} \cdot \underline{\boldsymbol{\nabla}}_{z} = 0, \; p \geq 0.$$

Or in components:

$$\sigma^{0}_{i3,3} = 0 \quad \text{and} \quad \sigma^{p}_{i\alpha,\alpha} + \sigma^{p+1}_{i3,3} = 0, \; p \geq 0.$$

Statics

The static boundary conditions on $\omega^{\pm}$ writes:

$$\underset{\sim}{\boldsymbol{\sigma}}^{p} \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = 0 \quad \text{when} \quad p \neq 2 \quad \text{and} \quad \underset{\sim}{\boldsymbol{\sigma}}^{2} \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = \frac{F_3}{2} \underline{\boldsymbol{e}}_3.$$

Or in components:

$$\sigma^{p}_{i3}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = 0 \quad \text{when} \quad p \neq 2$$

$$\sigma^{2}_{\alpha 3}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = 0 \quad \text{and} \quad \sigma^{2}_{33}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = \pm\frac{1}{2} F_3\left(Y_1, Y_2\right)$$

Kinematics

The non-dimensional displacement field is:

$$\underline{\boldsymbol{u}} = \eta^{-1}\underline{\boldsymbol{u}}^{-1} + \eta^{0}\underline{\boldsymbol{u}}^{0} + \eta^{1}\underline{\boldsymbol{u}}^{1} + \cdots$$

The non-dimensional strain field is:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{u}} \otimes^s \underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)} = \eta^{-2}\underset{\sim}{\boldsymbol{\varepsilon}}^{-2} + \eta^{-1}\underset{\sim}{\boldsymbol{\varepsilon}}^{-1} + \eta^{0}\underset{\sim}{\boldsymbol{\varepsilon}}^{0} + \cdots$$

with:

$$\underset{\sim}{\boldsymbol{\varepsilon}}^{-2} = \underline{\boldsymbol{u}}^{-1} \otimes^s \underline{\boldsymbol{\nabla}}_z \quad \text{and} \quad \underset{\sim}{\boldsymbol{\varepsilon}}^{p} = \underline{\boldsymbol{u}}^{p+1} \otimes^s \underline{\boldsymbol{\nabla}}_z + \underline{\boldsymbol{u}}^{p} \otimes^s \underline{\boldsymbol{\nabla}}_Y, \quad p \geq -1$$

In components:

$$\varepsilon^{-2}_{\alpha\beta} = 0, \quad \varepsilon^{-2}_{\alpha 3} = \frac{1}{2} u^{-1}_{\alpha,3} \quad \text{and} \quad \varepsilon^{-2}_{33} = u^{-1}_{3,3}$$

and for all $p \geq -1$:

$$\varepsilon^{p}_{\alpha\beta} = \frac{1}{2}\left(u^{p}_{\alpha,\beta} + u^{p}_{\beta,\alpha}\right), \quad \varepsilon^{p}_{\alpha 3} = \frac{1}{2}\left(u^{p+1}_{\alpha,3} + u^{p}_{3,\alpha}\right) \quad \text{and} \quad \varepsilon^{p}_{33} = u^{p+1}_{3,3}$$

Kinematics

The kinematic condition on the lateral boundary leads to:

$$\forall p \geq -1 \quad \text{and} \quad \forall (Y_1, Y_2) \in \partial\omega, \quad \underline{\boldsymbol{u}}^p (Y_1, Y_2) = 0.$$

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Lower order displacements

We set $\underset{\sim}{\varepsilon}^{-2} = 0$ which leads to:

$$\varepsilon^{-2}_{\alpha\beta} = 0, \quad \varepsilon^{-2}_{\alpha 3} = \frac{1}{2} u^{-1}_{\alpha,3} = 0 \quad \text{and} \quad \varepsilon^{-2}_{33} = u^{-1}_{3,3} = 0$$

Hence, $\underline{\boldsymbol{u}}^{-1}$ is a function of (Y_1, Y_2).
Moreover, u^{-1}_{α} is null since $\eta = -1$ is odd:

$$\underline{\boldsymbol{u}}^{-1} = U^{-1}_3 (Y_1, Y_2)\, \underline{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ U^{-1}_3 \end{pmatrix}$$

with the boundary conditions:

$$U^{-1}_3 = 0 \quad \forall (Y_1, Y_2) \in \partial\omega$$

Lower order displacements

We set $\underset{\approx}{\varepsilon}^{-1} = 0$ which leads to:

$$\varepsilon^{-1}_{\alpha\beta} = \frac{1}{2}\left(u^{-1}_{\alpha,\beta} + u^{-1}_{\beta,\alpha}\right) = 0, \quad \varepsilon^{-1}_{\alpha 3} = \frac{1}{2}\left(u^{0}_{\alpha,3} + U^{-1}_{3,\alpha}\right) = 0, \; \varepsilon^{-1}_{33} = u^{0}_{3,3} = 0$$

Hence, $\underline{\boldsymbol{u}}^0$ has the following form:

$$\underline{\boldsymbol{u}}^0 = -zU_3^{-1} \otimes \underline{\boldsymbol{\nabla}}_Y = \begin{pmatrix} -zU^{-1}_{3,1} \\ -zU^{-1}_{3,2} \\ 0 \end{pmatrix}$$

with the boundary conditions:

$$U^{-1}_{3,\alpha} n_\alpha = 0 \;\; \forall (Y_1, Y_2) \in \partial\omega$$

where $\underline{\boldsymbol{n}}$ is the outer normal to $\partial\omega$.

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Zeroth-order auxiliary problem

Equilibrium equation of order -1, compatibility equation, boundary conditions and constitutive equations of order 0 lead to (for $z \in [-\frac{1}{2}, \frac{1}{2}]$):

$$\begin{cases} \underset{\sim}{\boldsymbol{\sigma}}^0 \cdot \underline{\boldsymbol{\nabla}}_z = 0. \\ \underset{\sim}{\boldsymbol{\sigma}}^0 = \underset{\approx}{\mathbb{C}}(z) : \underset{\sim}{\boldsymbol{\varepsilon}}^0. \\ \underset{\sim}{\boldsymbol{\varepsilon}}^0 = \underline{\boldsymbol{u}}^1 \otimes^s \underline{\boldsymbol{\nabla}}_z + \underline{\boldsymbol{u}}^0 \otimes^s \underline{\boldsymbol{\nabla}}_Y. \\ \underset{\sim}{\boldsymbol{\sigma}}^0 \left(z = \pm\frac{1}{2}\right) \cdot \pm \underline{\boldsymbol{e}}_3 = 0 \end{cases}$$

$$\begin{cases} \sigma^0_{i3,3} = 0 \\ \sigma^0_{ij} = \mathbb{C}_{ijkl}\varepsilon^0_{kl} \\ \varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta} \\ \varepsilon^0_{\alpha 3} = \frac{1}{2}u^1_{\alpha,3} \quad \text{and} \quad \varepsilon^0_{33} = u^1_{3,3} \\ \sigma^0_{i3}\left(z = \pm\frac{1}{2}\right) = 0 \end{cases}$$

The lowest-order curvature is:

$$\underset{\sim}{\boldsymbol{K}}^{-1} := -U_3^{-1}\underline{\boldsymbol{\nabla}}_Y \otimes \underline{\boldsymbol{\nabla}}_Y \quad \text{or} \quad K^{-1}_{\alpha\beta} := -U^{-1}_{3,\alpha\beta}$$

Resolution

From

$$\sigma^0_{i3,3} = 0 \quad \text{and} \quad \sigma^0_{i3}\left(z = \pm\frac{1}{2}\right) = 0$$

we obtain plane-stress:

$$\sigma^0_{i3} = 0$$

The constitutive equation writes:

$$\begin{bmatrix} \sigma^0_{11} \\ \sigma^0_{22} \\ \sigma^0_{12} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1112} & 0 & 0 & \mathbb{C}_{1133} \\ \mathbb{C}_{1122} & \mathbb{C}_{2222} & \mathbb{C}_{2212} & 0 & 0 & \mathbb{C}_{2233} \\ \mathbb{C}_{1112} & \mathbb{C}_{2212} & \mathbb{C}_{1212} & 0 & 0 & \mathbb{C}_{1233} \\ 0 & 0 & 0 & \mathbb{C}_{1313} & \mathbb{C}_{1323} & 0 \\ 0 & 0 & 0 & \mathbb{C}_{1323} & \mathbb{C}_{2323} & 0 \\ \mathbb{C}_{1133} & \mathbb{C}_{2233} & \mathbb{C}_{12331} & 0 & 0 & \mathbb{C}_{3333} \end{bmatrix} \begin{bmatrix} zK^{-1}_{11} \\ zK^{-1}_{22} \\ 2zK^{-1}_{12} \\ 2\varepsilon^0_{13} \\ 2\varepsilon^0_{23} \\ \varepsilon^0_{33} \end{bmatrix}$$

Resolution

The strain is given by:

$$\varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta}(Y_1, Y_2), \quad \varepsilon^0_{\alpha 3} = 0 \quad \text{and} \quad \varepsilon^0_{33} = -\frac{z\mathbb{C}_{33\alpha\beta}(z)}{\mathbb{C}_{3333}(z)} K^{-1}_{\alpha\beta}(Y_1, Y_2)$$

The stress is given by:

$$\underset{\sim}{\boldsymbol{\sigma}}^0 = \underset{\approx}{\mathbb{s}}^K(z) : \underset{\sim}{\boldsymbol{K}}^{-1}(Y_1, Y_2) \quad \text{or in components} \quad \sigma^0_{ij} = \mathbb{s}^K_{ij\gamma\delta} K^{-1}_{\delta\gamma}$$

where the fourth-order stress localization tensor is:

$$\mathbb{s}^K_{\alpha\beta\gamma\delta}(z) := z\mathbb{C}^\sigma_{\alpha\beta\gamma\delta}(z) \quad \text{and} \quad \mathbb{s}^K_{i3\gamma\delta} := 0$$

and

$$\mathbb{C}^\sigma_{\alpha\beta\gamma\delta} = \mathbb{C}_{\alpha\beta\gamma\delta} - \mathbb{C}_{\alpha\beta 33}\mathbb{C}_{33\gamma\delta}/\mathbb{C}_{3333}$$

denotes the plane-stress elasticity tensor.

Resolution

By integrating of

$$\varepsilon^0_{\alpha 3} = 0 = \frac{1}{2} u^1_{\alpha,3} \quad \text{and} \quad \varepsilon^0_{33} = -\frac{z\mathbb{C}_{33\alpha\beta}}{\mathbb{C}_{3333}} K^{-1}_{\alpha\beta} = u^1_{3,3}$$

We find:

$$\underline{\boldsymbol{u}}^1 = \underset{\approx}{\mathfrak{u}}^K : \underset{\sim}{\boldsymbol{K}}^{-1} + U^1_3 \underline{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ \mathfrak{u}^K_{3\alpha\beta} K^{-1}_{\beta\alpha} + U^1_3 \end{pmatrix}.$$

where the displacement localization tensor $\underset{\approx}{\mathfrak{u}}^K(z)$ related to the curvature is given by:

$$\mathfrak{u}^K_{3\alpha\beta}(z) := -\left[\int_{-\frac{1}{2}}^{z} r \frac{\mathbb{C}_{33\alpha\beta}}{\mathbb{C}_{3333}} dr\right]^* \quad \text{and} \quad \mathfrak{u}^K_{\alpha\beta\gamma} := 0$$

where $[\bullet]^*$ denotes the averaged-out distribution: $[\bullet]^* := \bullet - \langle \bullet \rangle$.

Isotropic materials

The leading order strain:

$$\varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta}, \quad \varepsilon^0_{\alpha 3} = 0 \quad \text{and} \quad \varepsilon^0_{33} = -\frac{\nu}{1-\nu} zK^{-1}_{\alpha\alpha}$$

The leading order stress is derived through $\underset{\approx}{\mathbb{C}}^\sigma$:

$$\begin{bmatrix} \sigma^0_{11} \\ \sigma^0_{22} \\ \sigma^0_{12} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ & \frac{E}{1-\nu^2} & 0 \\ \text{SYM} & & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} zK^{-1}_{11} \\ zK^{-1}_{22} \\ 2zK^{-1}_{12} \end{bmatrix}$$

The displacement corrector is:

$$\underline{\boldsymbol{u}}^1 = \underset{\approx}{\underline{\mathfrak{u}}}^K : \underset{\sim}{\boldsymbol{K}}^{-1} + U^1_3 \underline{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\nu}{2(1-\nu)} \left(\frac{1}{12} - z^2 \right) K^{-1}_{\alpha\alpha} + U^1_3 \end{pmatrix}.$$

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Determination of U_3^{-1}

Resultants

The zeroth-order bending moment is defined as

$$M^0_{\alpha\beta}\left(Y_1, Y_2\right) := \left\langle z\sigma^0_{\alpha\beta}\right\rangle ,$$

The first-order shear force is:

$$Q^1_{\alpha}\left(Y_1, Y_2\right) := \left\langle \sigma^1_{3\alpha}\right\rangle .$$

Determination of U_3^{-1}

Equilibrium

Recall that:

$$\sigma^0_{i3,3} = 0 \quad \text{and} \quad \sigma^p_{i\alpha,\alpha} + \sigma^{p+1}_{i3,3} = 0, \; p \geq 0.$$

The bending equilibrium equations are:

$$\left\langle z \left(\sigma^0_{\alpha\beta,\beta} + \sigma^1_{\alpha 3,3} \right) \right\rangle = 0 = M^0_{\alpha\beta,\beta} - Q^1_\alpha$$

The out-of-plane equilibrium equation is:

$$\left\langle \sigma^1_{3\alpha,\alpha} + \sigma^2_{33,3} \right\rangle = 0 = Q^1_{\alpha,\alpha} + F_3$$

Finally, we have:

$$M^0_{\alpha\beta,\beta\alpha} + F_3 = 0$$

Determination of U_3^{-1}

Constitutive equation

From

$$\left\langle z\underset{\sim}{\boldsymbol{\sigma}}^0 \right\rangle = \left\langle z\underset{\approx}{\mathbf{s}}^K(z) : \underset{\sim}{\boldsymbol{K}}^{-1} \right\rangle = \left\langle z^2 \underset{\approx}{\mathbb{C}}^\sigma(z) : \underset{\sim}{\boldsymbol{K}}^{-1} \right\rangle$$

we obtain the Kirchhoff's constitutive equation:

$$\underset{\sim}{\boldsymbol{M}}^0 = \underset{\approx}{\boldsymbol{D}} : \underset{\sim}{\boldsymbol{K}}^{-1} \quad \text{where:} \quad \underset{\approx}{\boldsymbol{D}} = \left\langle z^2 \, \underset{\approx}{\mathbb{C}}^\sigma \right\rangle$$

The Kirchhoff-Love plate equations are:

$$\begin{cases} \underset{\sim}{\boldsymbol{M}}^0 : \left(\underline{\boldsymbol{\nabla}}_Y \otimes \underline{\boldsymbol{\nabla}}_Y \right) + F_3 = 0, \quad \text{on} \quad \omega \\ \underset{\sim}{\boldsymbol{M}}^0 = \underset{\approx}{\boldsymbol{D}} : \underset{\sim}{\boldsymbol{K}}^{-1}, \quad \text{on} \quad \omega \\ \underset{\sim}{\boldsymbol{K}}^{-1} = -U_3^{-1} \underline{\boldsymbol{\nabla}}_Y \otimes \underline{\boldsymbol{\nabla}}_Y, \quad \text{on} \quad \omega \\ U_3^{-1} = 0 \quad \text{and} \quad \left(U_3^{-1} \otimes \underline{\boldsymbol{\nabla}}_Y \right) \cdot \underline{\boldsymbol{n}} = 0 \quad \text{on} \quad \partial\omega \end{cases}$$

Summary of the Kirchhoff-Love model

The displacement is approximated by:

$$\underline{\boldsymbol{u}} \approx \eta^{-1}\underline{\boldsymbol{u}}^{-1} + \underline{\boldsymbol{u}}^{0} = \begin{pmatrix} -zU_{3,1}^{-1} \\ -zU_{3,2}^{-1} \\ \eta^{-1}U_3^{-1} \end{pmatrix} = \underline{\boldsymbol{u}}^{LK}$$

The strain $\underset{\approx}{\varepsilon}$ is approximated by $\underset{\approx}{\varepsilon}^0 \neq \underline{\boldsymbol{u}}^{LK} \otimes^s \underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)}$ with:

$$\varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta}(Y_1, Y_2), \quad \varepsilon^0_{\alpha 3} = 0 \quad \text{and} \quad \varepsilon^0_{33} = -\frac{z\mathbb{C}_{33\alpha\beta}(z)}{\mathbb{C}_{3333}(z)}K^{-1}_{\alpha\beta}(Y_1, Y_2)$$

where

$$K^{-1}_{\alpha\beta} = -U^{-1}_{3,\alpha\beta}$$

The stress $\underset{\approx}{\sigma}$ is approximated by $\underset{\approx}{\sigma}^0$ such that $\underset{\approx}{\sigma}^0 \cdot \underline{\boldsymbol{\nabla}}^{\eta}_{(Y,z)} \neq 0$ and

$$\sigma^0_{\alpha\beta} = z\mathbb{C}^{\sigma}_{\alpha\beta\gamma\delta}(z)\,K^{-1}_{\delta\gamma} \quad \text{and} \quad \sigma^0_{i3} = 0$$

Lateral boundary conditions

It should be emphasized that the assumed expansion is not compatible with clamped lateral boundary conditions. Indeed,

$$\underline{\boldsymbol{u}}^1 = \underset{\approx}{\underline{\mathbb{u}}}^K : \underset{\sim}{\boldsymbol{K}}^{-1} + U_3^1 \underline{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ \mathbb{u}^K_{3\alpha\beta} K^{-1}_{\beta\alpha} + U_3^1 \end{pmatrix} \neq 0.$$

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First-order auxiliary problem

Equilibrium equation for order 0, compatibility equation, boundary conditions and constitutive equations of order 1 lead to (for $z \in [-\frac{1}{2}, \frac{1}{2}]$):

$$\begin{cases} \underset{\sim}{\sigma}^0 \cdot \underline{\nabla}_Y + \underset{\sim}{\sigma}^1 \cdot \underline{\nabla}_z = 0 \\ \underset{\sim}{\sigma}^1 = \underset{\approx}{\mathbb{C}}(z) : \underset{\sim}{\varepsilon}^1 \\ \underset{\sim}{\varepsilon}^1 = \underline{\boldsymbol{u}}^2 \otimes^s \underline{\nabla}_z + \underline{\boldsymbol{u}}^1 \otimes^s \underline{\nabla}_Y \\ \underset{\sim}{\sigma}^1 \left(z = \pm\frac{1}{2}\right) \cdot \pm\underline{\boldsymbol{e}}_3 = 0 \end{cases}$$

$$\begin{cases} \sigma^0_{i\alpha,\alpha} + \sigma^1_{i3,3} = 0 \\ \sigma^1_{ij} = \mathbb{C}_{ijkl}\varepsilon^1_{kl} \\ \varepsilon^1_{\alpha\beta} = u^1_{(\alpha,\beta)} = 0 \\ \varepsilon^1_{\alpha 3} = \frac{1}{2}\left(u^2_{\alpha,3} + u^1_{3,\alpha}\right) \\ \varepsilon^1_{33} = u^2_{3,3} = 0 \\ \sigma^1_{i3}\left(z = \pm\frac{1}{2}\right) = 0 \end{cases}$$

Resolution

Transverse stress

From

$$\begin{cases} \sigma^0_{\beta\alpha,\alpha} + \sigma^1_{\beta 3,3} = 0 \\ \sigma^0_{\beta\alpha,\alpha} = \left(\mathbb{s}^K_{\beta\alpha\gamma\delta}(z)\, K^{-1}_{\delta\gamma} \right)_{,\alpha} = z \mathbb{C}^\sigma_{\beta\alpha\gamma\delta}(z)\, K^{-1}_{\delta\gamma,\alpha} \\ \sigma^1_{\beta 3}\left(z = \pm\frac{1}{2}\right) = 0 \end{cases}$$

we obtain the first-order transverse shear stress:

$$\sigma^1_{\alpha 3} = -\int_{-\frac{1}{2}}^{z} r\, \mathbb{C}^\sigma_{\alpha\beta\gamma\delta}\, dr\, K^{-1}_{\delta\gamma,\beta}$$

Resolution

In-plane stress

From $\varepsilon^1_{\alpha\beta} = 0$ and $\varepsilon^1_{33} = 0$ and the constitutive equation:

$$\begin{bmatrix} \sigma^1_{11} \\ \sigma^1_{22} \\ \sigma^1_{12} \\ \sigma^1_{13} \\ \sigma^1_{23} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1112} & 0 & 0 & \mathbb{C}_{1133} \\ & \mathbb{C}_{2222} & \mathbb{C}_{2212} & 0 & 0 & \mathbb{C}_{2233} \\ & & \mathbb{C}_{1212} & 0 & 0 & \mathbb{C}_{1233} \\ & & & \mathbb{C}_{1313} & \mathbb{C}_{1323} & 0 \\ & \text{SYM} & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{3333} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\varepsilon^1_{13} \\ 2\varepsilon^1_{23} \\ 0 \end{bmatrix}$$

we have $\sigma^1_{\alpha\beta} = 0$ and the first-order stress localization writes as:

$$\underset{\sim}{\boldsymbol{\sigma}}^1 = \underset{\approx}{\mathbb{s}}^{K\nabla}(z) \vdots \left(\underset{\sim}{\boldsymbol{K}}^{-1} \otimes \underline{\boldsymbol{\nabla}}_Y \right)$$

where we defined the fifth-order localization tensor as:

$$\mathbb{s}^{K\nabla}_{\alpha\beta\gamma\delta\eta} := 0, \quad \mathbb{s}^{K\nabla}_{\alpha3\gamma\delta\eta}(z) := -\int_{-\frac{1}{2}}^{z} r\, \mathbb{C}^{\sigma}_{\alpha\gamma\delta\eta}\, dr \quad \text{and} \quad \mathbb{s}^{K\nabla}_{33\gamma\delta\eta} := 0$$

Resolution

Displacement

We find that the second-order displacement field writes as:

$$\underline{\boldsymbol{u}}^2 = \underset{\frown}{\underline{\mathbb{u}}}^{K\nabla}(z) \vdots \left(\underset{\sim}{\boldsymbol{K}}^{-1} \otimes \underline{\boldsymbol{\nabla}}_Y\right) - zU_3^1 \otimes \underline{\boldsymbol{\nabla}}_Y = \begin{pmatrix} -zU^1_{3,1} + \mathbb{u}^{K\nabla}_{1\beta\gamma\delta} K^{-1}_{\delta\gamma,\beta} \\ -zU^1_{3,2} + \mathbb{u}^{K\nabla}_{2\beta\gamma\delta} K^{-1}_{\delta\gamma,\beta} \\ 0 \end{pmatrix}$$

where the displacement localization tensor related to the curvature gradient writes as:

$$\mathbb{u}^{K\nabla}_{\alpha\beta\gamma\delta}(z) := -\int_0^z \left(4\mathbb{S}_{\alpha 3\eta 3} \int_{-\frac{1}{2}}^{y} v\, \mathbb{C}^{\sigma}_{\eta\beta\gamma\delta}\, dv + \delta_{\alpha\beta}\mathbb{u}^{K}_{3\gamma\delta} \right) dy$$

and

$$\mathbb{u}^{K\nabla}_{3\beta\gamma\delta} := 0$$

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The general form of the expansion

It can be formally shown that we have:

$$\underline{\boldsymbol{u}} = \frac{U_3}{\eta}\underline{\boldsymbol{e}}_3 - zU_3 \otimes \underline{\boldsymbol{\nabla}}_Y + \eta\ \underset{\approx}{\mathfrak{u}}^K : \underset{\sim}{\boldsymbol{K}} + \eta^2\ \underset{\approx}{\mathfrak{u}}^{K\nabla} \vdots \underset{\sim}{\boldsymbol{K}} \otimes \underline{\boldsymbol{\nabla}}_Y + \dots$$

where

$$U_3 := \sum_{p=-1}^{\infty} \eta^{p+1} U_3^p = \eta \langle u_3 \rangle \quad \text{and} \quad \underset{\sim}{\boldsymbol{K}} := -U_3 \underline{\boldsymbol{\nabla}}_Y \otimes \underline{\boldsymbol{\nabla}}_Y$$

We have also for the stress:

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\mathfrak{s}}^K : \underset{\sim}{\boldsymbol{K}} + \eta\ \underset{\approx}{\mathfrak{s}}^{K\nabla} \vdots \underset{\sim}{\boldsymbol{K}} \otimes \underline{\boldsymbol{\nabla}}_Y + \dots$$

A higher order plate model from asymptotic expansions?

Including shear effects...:

- ... from asymptotic expansion?:
  $U_3 \in \mathcal{C}^6(\omega)$
- ... from the approach from Smyshlyaev and Cherednichenko (2000)?:
  $U_3 \in \mathcal{C}^4(\omega)$
- ... with the Bending-Gradient theory:
  $U_3 \in \mathcal{C}^1(\omega)$

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Stress localization as function of static variables

The stress field can be accurately approximated by:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}} = \underset{\approx}{\mathbf{s}}^{K} : \underset{\sim}{\boldsymbol{\chi}} + \eta\, \underset{\approx}{\mathbf{s}}^{K\nabla} \vdots \underset{\sim}{\boldsymbol{\chi}} \otimes \underline{\boldsymbol{\nabla}}_{Y}$$

$\underset{\sim}{\boldsymbol{\chi}} = (\chi_{\alpha\beta})(Y_1, Y_2)$ is an unknown symmetric second-order tensor field.

Choice of $\underset{\sim}{\boldsymbol{\chi}}$?: The minimum of complementary energy!

The corresponding bending moment is:

$$\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} = \underset{\approx}{\boldsymbol{D}} : \underset{\sim}{\boldsymbol{\chi}} \quad \text{where} \quad \underset{\approx}{\boldsymbol{D}} = \left\langle z^2\, \underset{\approx}{\mathbb{C}}^{\sigma} \right\rangle \quad \text{and} \quad \underset{\approx}{\boldsymbol{d}} = \underset{\approx}{\boldsymbol{D}}^{-1}$$

Its gradient is:

$$\underset{\frown}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \otimes \underline{\boldsymbol{\nabla}}_{Y} \quad \text{or} \quad R_{\alpha\beta\gamma} = M^{\mathrm{BG}}_{\alpha\beta,\gamma} \quad \text{with} \quad R_{\alpha\beta\gamma} = R_{\beta\alpha\gamma}$$

Stress localization as function of static variables

It is possible to rewrite $\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}}$ in terms of $\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}$ and $\underset{\frown}{\boldsymbol{R}}$:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}} = \underset{\approx}{\mathbf{s}}^{K} : \left(\underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}\right) + \eta\, \underset{\approx}{\mathbf{s}}^{K\nabla} \vdots \left(\underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}\right) \otimes \underline{\boldsymbol{\nabla}}_{Y}$$

and

$$\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}} = \underset{\approx}{\mathbf{s}}^{M} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} + \eta\, \underset{\approx}{\mathbf{s}}^{R} \vdots \underset{\frown}{\boldsymbol{R}}$$

where the localizations tensors are given by:

$$\underset{\approx}{\mathbf{s}}^{M} = \underset{\approx}{\mathbf{s}}^{K} : \underset{\approx}{\boldsymbol{d}}, \quad \underset{\approx}{\mathbf{s}}^{R} = \underset{\approx}{\mathbf{s}}^{K\nabla} : \underset{\approx}{\boldsymbol{d}}$$

The Bending-Gradient stress energy

Plugging $\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}}$ into the complementary energy of the full 3D problem leads to the following functional:

$$P^{*BG}\left(\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}, \underset{\frown}{\boldsymbol{R}}\right) = \int_\omega w^{*KL}\left(\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}\right) + \eta^2 w^{*BG}\left(\underset{\frown}{\boldsymbol{R}}\right) d\omega$$

where the stress elastic energies are defined as:

$$w^{*KL}\left(\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}\right) = \frac{1}{2}\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} : \underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \quad \text{and} \quad w^{*BG}\left(\underset{\frown}{\boldsymbol{R}}\right) = \frac{1}{2}{}^{T}\underset{\frown}{\boldsymbol{R}} \vdots \underset{\frown\frown}{\boldsymbol{h}} \vdots \underset{\frown}{\boldsymbol{R}}$$

with:

$$\underset{\frown\frown}{\boldsymbol{h}} = \left\langle {}^{T}\underset{\approx}{\mathbb{s}}^{R} : \underset{\approx}{\mathbb{S}} : \underset{\approx}{\mathbb{s}}^{R} \right\rangle$$

This sixth-order tensor is the compliance related to the transverse shear of the plate. <u>It is positive, symmetric, but not definite in the general case.</u>

Extended plate equilibrium equations

Exact plate equilibrium equations

The total bending moment and the total shear force are defined as

$$M_{\alpha\beta}(Y_1, Y_2) = \langle z\sigma_{\alpha\beta}\rangle, \quad \text{and} \quad Q_\alpha(Y_1, Y_2) = \eta^{-1}\langle\sigma_{3\alpha}\rangle.$$

Moment equilibrium equations:

$$\left\langle z\left(\sigma_{\alpha\beta,\beta} + \eta^{-1}\sigma_{\alpha 3,3}\right)\right\rangle = M_{\alpha\beta,\beta} - Q_\alpha = 0 \quad \text{or} \quad \underset{\sim}{\boldsymbol{M}} \cdot \underline{\boldsymbol{\nabla}}_Y - \underline{\boldsymbol{Q}} = 0$$

The out-of-plane equilibrium equation:

$$\eta^{-1}\left\langle\sigma_{3\alpha,\alpha} + \eta^{-1}\sigma_{33,3}\right\rangle = Q_{\alpha,\alpha} + F_3 = 0 \quad \text{or} \quad \underline{\boldsymbol{Q}} \cdot \underline{\boldsymbol{\nabla}}_Y + F_3 = 0$$

Extended plate equilibrium equations

Link between shear forces and generalized shear forces

$\underline{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{M}} \cdot \underline{\boldsymbol{\nabla}}_Y$ is now replaced by $\underset{\frown}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \otimes \underline{\boldsymbol{\nabla}}_Y$

We have the following relation:

$$\underset{\approx}{\boldsymbol{i}} \,\vdots\, \underset{\frown}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \cdot \underline{\boldsymbol{\nabla}}_Y = \underline{\boldsymbol{Q}}^{\mathrm{BG}} \quad \text{or} \quad R_{\alpha\beta\beta} = M^{\mathrm{BG}}_{\alpha\beta,\beta} = Q^{\mathrm{BG}}_{\alpha}$$

where

$$i_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

Mechanical meaning of $\underset{\frown}{\boldsymbol{R}}$

$$Q_\alpha = R_{\alpha\beta\beta} \Leftrightarrow \left\{ \begin{array}{l} Q_1 = R_{111} + R_{122} = M_{11,1} + M_{12,2} \\ Q_2 = R_{121} + R_{222} = M_{21,1} + M_{22,2} \end{array} \right.$$

The Bending-Gradient statically compatible fields

The Bending-Gradient stress energy must be minimized over the set:

$$SC^{\mathrm{BG}}: \begin{cases} \underset{\frown}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \otimes \underline{\boldsymbol{\nabla}}_Y & \text{or} \quad R_{\alpha\beta\gamma} = M^{\mathrm{BG}}_{\alpha\beta,\gamma} \\ \left(\underset{\approx}{\boldsymbol{i}} \,\vdots\, \underset{\frown}{\boldsymbol{R}}\right) \cdot \underline{\boldsymbol{\nabla}}_Y + F_3 = 0 & \text{or} \quad R_{\alpha\beta\beta,\alpha} + F_3 = 0 \end{cases}$$

The Bending-Gradient constitutive equations

Now we define the generalized strains as:

$$\underset{\sim}{\chi} = \frac{\partial w^{*KL}}{\partial \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}} \quad \text{and} \quad \underset{\frown}{\boldsymbol{\Gamma}} = \frac{\partial w^{*BG}}{\partial \underset{\frown}{\boldsymbol{R}}}$$

Note that the third-order tensor has the symmetry:

$$\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$$

This leads to the following constitutive equations:

$$\begin{cases} \underset{\sim}{\chi} = \underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} & \text{or} \quad \chi_{\alpha\beta} = d_{\alpha\beta\gamma\delta} M^{\mathrm{BG}}_{\delta\gamma} \\ \underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown\frown}{\boldsymbol{h}} \vdots \underset{\frown}{\boldsymbol{R}} & \text{or} \quad \Gamma_{\alpha\beta\gamma} = h_{\alpha\beta\gamma\delta\mu\nu} R_{\nu\mu\delta} \end{cases}$$

Dualization of equilibrium equations

Multiplying $R_{\alpha\beta\gamma} = M^{\mathsf{BG}}_{\alpha\beta,\gamma}$ with $\Phi_{\alpha\beta\gamma}$ and integrating by parts on the plate domain ω yield:

$$\int_\omega M^{\mathsf{BG}}_{\alpha\beta}\Phi_{\alpha\beta\gamma,\gamma} + R_{\alpha\beta\gamma}\Phi_{\alpha\beta\gamma}d\omega = \int_{\partial\omega} M^{\mathsf{BG}}_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma dl$$

Multiplying $R_{\alpha\beta\beta,\alpha} + F_3 = 0$ with U^{BG}_3 and integrating by parts on the plate domain ω yield:

$$\int_\omega R_{\alpha\beta\beta}U^{\mathsf{BG}}_{3,\alpha}d\omega = \int_{\partial\omega} R_{\alpha\beta\beta}n_\alpha U^{\mathsf{BG}}_3 dl + \int_\omega F_3 U^{\mathsf{BG}}_3 d\omega$$

Weak formulation

Adding these equations leads to the following expression:

$$\int_\omega M^{BG}_{\alpha\beta}\Phi_{\alpha\beta\gamma,\gamma} + R_{\alpha\beta\gamma}\left(\Phi_{\alpha\beta\gamma} + \frac{1}{2}\left(\delta_{\beta\gamma}U^{BG}_{3,\alpha} + \delta_{\alpha\gamma}U^{BG}_{3,\beta}\right)\right) d\omega = \int_\omega F_3 U^{BG}_3 d\omega + \int_{\partial\omega} M^{BG}_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma + R_{\alpha\beta\beta}n_\alpha U^{BG}_3 dl$$

Therefore, we have obtained the weak formulation of this plate theory:

$$V^{BG}_{int} = V^{BG}_{ext}$$

where

$$V^{BG}_{int} = \int_\omega \underset{\approx}{\boldsymbol{M}}^{BG} : \left(\underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{\nabla}}_Y\right) + {}^T\underset{\frown}{\boldsymbol{R}} \vdots \left(\underset{\frown}{\boldsymbol{\Phi}} + \underset{\approx}{\boldsymbol{i}} \cdot \underline{\boldsymbol{\nabla}}_Y U^{BG}_3\right) d\omega$$

$$V^{BG}_{ext} = \int_\omega F_3 U^{BG}_3 d\omega + \int_{\partial\omega} \underset{\approx}{\boldsymbol{M}}^{BG} : \left(\underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{n}}\right) + \left(\underset{\approx}{\boldsymbol{i}} \vdots \underset{\frown}{\boldsymbol{R}} \cdot \underline{\boldsymbol{n}}\right) U^{BG}_3 dl$$

and $\underline{\boldsymbol{n}}$ is the in-plane unit vector outwardly normal to ω.

Kinematic compatibility conditions

We identify the internal power obtained by dualization

$$V_{int}^{BG} = \int_\omega \underset{\sim}{\boldsymbol{M}}^{BG} : \left(\underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{\nabla}}_Y \right) + {}^T\underset{\frown}{\boldsymbol{R}} \vdots \left(\underset{\frown}{\boldsymbol{\Phi}} + \underset{\approx}{\boldsymbol{i}} \cdot \underline{\boldsymbol{\nabla}}_Y U_3^{BG} \right) d\omega$$

with the one obtained with the constitutive equations

$$V_{int}^{BG} = \int_\omega \underset{\sim}{\boldsymbol{M}}^{BG} : \underset{\sim}{\chi} + \eta^2 \, {}^T\underset{\frown}{\boldsymbol{R}} \vdots \underset{\frown}{\boldsymbol{\Gamma}} d\omega$$

Finally, we define the set of kinematically compatible fields as

$$KC^{BG} : \begin{cases} \underset{\sim}{\chi} = \underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{\nabla}}_Y \\ \eta^2 \underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown}{\boldsymbol{\Phi}} + \underset{\approx}{\boldsymbol{i}} \cdot \underline{\boldsymbol{\nabla}}_Y U_3^{BG} \end{cases}$$

to which the following boundary conditions must be added for a clamped plate:

$$U_3^{BG} = 0 \quad \text{and} \quad \underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{n}} = 0 \text{ on } \partial\omega$$

Summary

The Bending-Gradient plate theory equations are the following:

$$\begin{cases} \underset{\frown}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \otimes \underline{\boldsymbol{\nabla}}_Y \quad \text{and} \quad \left(\underset{\approx}{\boldsymbol{i}} \vdots \underset{\frown}{\boldsymbol{R}}\right) \cdot \underline{\boldsymbol{\nabla}}_Y + F_3 = 0 \quad \text{on} \quad \omega \\ \underset{\sim}{\chi} = \underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \quad \text{and} \quad \underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown\!\!\frown}{\boldsymbol{h}} \vdots \underset{\frown}{\boldsymbol{R}} \quad \text{on} \quad \omega \\ \underset{\sim}{\chi} = \underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{\nabla}}_Y \quad \text{and} \quad \eta^2 \underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown}{\boldsymbol{\Phi}} + \underset{\approx}{\boldsymbol{i}} \cdot \underline{\boldsymbol{\nabla}}_Y U_3^{\mathrm{BG}} \quad \text{on} \quad \omega \\ U_3^{\mathrm{BG}} = 0 \quad \text{and} \quad \underset{\frown}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{n}} = 0 \text{ on } \partial\omega \end{cases}$$

Note that:

$$\underset{\sim}{\chi} = \underset{\sim}{\boldsymbol{K}}^{\mathrm{BG}} + \eta^2 \underset{\frown}{\boldsymbol{\Gamma}} \cdot \underline{\boldsymbol{\nabla}}_Y \quad \text{where} \quad \underset{\sim}{\boldsymbol{K}}^{\mathrm{BG}} = -U_3^{\mathrm{BG}} \underline{\boldsymbol{\nabla}}_Y \otimes \underline{\boldsymbol{\nabla}}_Y$$

Setting $\eta^2 = 0$ in the Bending-Gradient model leads exactly to Kirchhoff-Love plate model.

3D localization

Once the exact solution of the macroscopic problem is derived, it is possible to reconstruct the local displacement field. We suggest the following 3D displacement field where U^{BG}, $\underset{\frown}{\mathbf{\Phi}}$ are the fields solution of the plate problem:

$$\underline{\boldsymbol{u}}^{BG} = \frac{U_3^{BG}}{\eta}\underline{\boldsymbol{e}}_3 - zU_3^{BG}\otimes\underline{\boldsymbol{\nabla}}_Y + \eta\,\underset{\approx}{\mathfrak{u}}^K : \underset{\approx}{\boldsymbol{\chi}} + \eta^2\,\underset{\frown}{\underline{\mathfrak{u}}}^{K\nabla} \vdots \left(\underset{\approx}{\boldsymbol{\chi}}\otimes\underline{\boldsymbol{\nabla}}_Y\right)$$

Defining the strain as

$$\underset{\approx}{\boldsymbol{\varepsilon}}^{BG} = \underset{\approx\approx}{\mathbb{S}} : \underset{\approx}{\boldsymbol{\sigma}}^{BG}$$

it is possible to check that:

$$\varepsilon\left(\underline{\boldsymbol{u}}^{BG}\right)_{(Y,z)} - \underset{\approx}{\boldsymbol{\varepsilon}}^{BG} = \eta^2\left(\left(\underset{\approx}{\boldsymbol{\delta}}\otimes^s\underset{\frown}{\underline{\mathfrak{u}}}^{K\nabla}\right) :: \left(\underset{\approx}{\boldsymbol{\chi}}\otimes\underline{\boldsymbol{\nabla}}_Y^2\right) + z\underset{\frown}{\boldsymbol{\Gamma}}\cdot\underline{\boldsymbol{\nabla}}_Y\right)$$

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Homogeneous plates

In this case, we have:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{BG}} = \begin{cases} \sigma^{\mathrm{BG}}_{\alpha\beta} & = 12z\, i_{\alpha\beta\gamma\delta} M^{\mathrm{BG}}_{\delta\gamma} & = 12z M^{\mathrm{BG}}_{\alpha\beta} \\ \sigma^{\mathrm{BG}}_{\alpha 3} & = \eta \frac{3}{2}\left(1 - 4z^2\right)\, i_{\alpha\beta\gamma\delta} R_{\delta\gamma\beta} & = \eta \frac{3}{2}\left(1 - 4z^2\right) Q^{\mathrm{BG}}_{\alpha} \\ \sigma^{\mathrm{BG}}_{33} & = 0 & \end{cases}$$

which is a function of $\underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}}$ and $\underline{\boldsymbol{Q}}^{\mathrm{BG}} = \underset{\approx}{\boldsymbol{i}} \,\vdots\, \underset{\frown}{\boldsymbol{R}}$ instead of the whole $\underset{\frown}{\boldsymbol{R}}$

The constitutive equations

The Bending-Gradient part of the stress energy becomes:

$$w^{*BG}\left(\underset{\frown}{\boldsymbol{R}}\right) = \frac{1}{2}{}^{T}\underset{\frown}{\boldsymbol{R}} \,\vdots\, \underset{\frown\!\frown}{\boldsymbol{h}} \,\vdots\, \underset{\frown}{\boldsymbol{R}} = \frac{1}{2}\underline{\boldsymbol{Q}}^{\mathrm{BG}} \cdot \underset{\sim}{\boldsymbol{h}}^{RM} \cdot \underline{\boldsymbol{Q}}^{\mathrm{BG}}$$

with:

$$\underset{\frown\!\frown}{\boldsymbol{h}} = \underset{\approx}{\boldsymbol{i}} \cdot \underset{\sim}{\boldsymbol{h}}^{RM} \cdot \underset{\approx}{\boldsymbol{i}}$$

where the Reissner's shear forces stiffness is given by:

$$h^{RM}_{\alpha\beta} = \frac{6}{5}\mathbb{S}_{\alpha3\beta3}$$

(it is equal to $\frac{6}{5G}\delta_{\alpha\beta}$ with G the shear modulus for isotropic plates). The Bending-Gradient constitutive equation becomes:

$$\underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown\!\frown}{\boldsymbol{h}} \,\vdots\, \underset{\frown}{\boldsymbol{R}} = \underset{\approx}{\boldsymbol{i}} \cdot \underline{\gamma}$$

with

$$\underline{\gamma} = \underset{\sim}{\boldsymbol{h}}^{RM} \cdot \underline{\boldsymbol{Q}}^{\mathrm{BG}}$$

The kinematics

Using the kinematic compatibility

$$\eta^2 \underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown}{\boldsymbol{\Phi}} + \underset{\approx}{\underline{\boldsymbol{i}}} \cdot \underline{\boldsymbol{\nabla}}_Y U_3^{\mathrm{BG}},$$

we find that $\underset{\frown}{\boldsymbol{\Phi}}$ is also of the form:

$$\underset{\frown}{\boldsymbol{\Phi}} = \underset{\approx}{\underline{\boldsymbol{i}}} \cdot \underline{\boldsymbol{\varphi}}$$

where $\underline{\boldsymbol{\varphi}}$ is the classical rotation vector of the Reissner theory. Therefore, the kinematic unknowns are U_3^{BG} and $\underline{\boldsymbol{\varphi}}$, and we have:

$$\left\{ \begin{array}{lclcl} \underset{\sim}{\boldsymbol{\chi}} & = & \underline{\boldsymbol{\varphi}} \otimes^s \underline{\boldsymbol{\nabla}}_Y & = & \underset{\approx}{\underline{\boldsymbol{d}}} : \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \\ \eta^2 \underline{\boldsymbol{\gamma}} & = & \underline{\boldsymbol{\varphi}} + U_3^{\mathrm{BG}} \otimes \underline{\boldsymbol{\nabla}}_Y & = & \eta^2 \underset{\sim}{\boldsymbol{h}}^{RM} \cdot \underline{\boldsymbol{Q}}^{\mathrm{BG}} \end{array} \right.$$

Static

The following boundary conditions must be added for a clamped plate:

$$U_3^{\mathrm{BG}} = 0 \quad \text{and} \quad \underline{\varphi} = 0 \text{ on } \partial\omega$$

Finally, the balance equations are:

$$\begin{cases} \underset{\sim}{\boldsymbol{M}}^{\mathrm{BG}} \cdot \underline{\boldsymbol{\nabla}}_Y - \underline{\boldsymbol{Q}}^{\mathrm{BG}} = 0 \text{ on } \omega \\ \underline{\boldsymbol{Q}}^{\mathrm{BG}} \cdot \underline{\boldsymbol{\nabla}}_Y + F_3 = 0 \text{ on } \omega \end{cases}$$

In conclusion: the Bending-Gradient theory completely coincides for homogeneous plates with the Reissner-Mindlin model.

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Distance between the BG and the RM models

We introduce the following relative distance:

$$\Delta^{\mathrm{RM/BG}} = \frac{\|\underset{\frown}{\boldsymbol{h}}^{\mathrm{W}}\|}{\|\underset{\frown}{\boldsymbol{h}}\|}$$

where

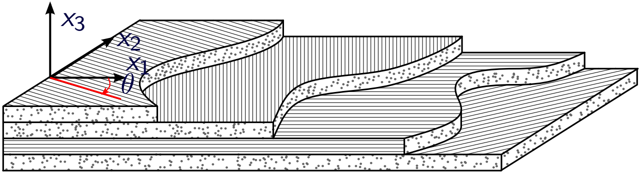
$$\|\underset{\frown}{\boldsymbol{h}}\| = \sqrt{{}^{T}\underset{\frown}{\boldsymbol{h}} \,\vdots\vdots\, \underset{\frown}{\boldsymbol{h}}}$$

is the norm for Bending-Gradient compliance tensors and $\underset{\frown}{\boldsymbol{h}}^{\mathrm{W}}$ is the pure warping part of $\underset{\frown}{\boldsymbol{h}}$:

$$\underset{\frown}{\boldsymbol{h}}^{\mathrm{W}} = \underset{\frown}{\boldsymbol{h}} - \frac{4}{9}\underset{\approx}{\boldsymbol{i}} \cdot \underset{\approx}{\boldsymbol{i}} \,\vdots\, \underset{\frown}{\boldsymbol{h}} \,\vdots\, \underset{\approx}{\boldsymbol{i}} \cdot \underset{\approx}{\boldsymbol{i}}$$

$\Delta^{\mathrm{RM/BG}}$ gives an estimate of the pure warping fraction of the shear stress energy. When the plate constitutive equation is restricted to a Reissner-Mindlin one we have exactly $\Delta^{\mathrm{RM/BG}} = 0$.

Distance between the BG and the RM models



Stack	$[0°]$	$[30°, -30°]_s$	$[0°, -45°, 90°, 45°]_s$
$\Delta^{RM/BG}$	0	16.0%	12.4%

Table: The criterion $\Delta^{RM/BG}$ for several laminates

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Voigt Notations

We introduce the linear operator $\left[\underset{\sim}{\bullet}\right]$ reallocating tensor components. For instance, the bending moment and the curvature are reallocated in a vector form:

$$\left[\underset{\sim}{\boldsymbol{M}}\right] = \begin{pmatrix} M_{11} \\ M_{22} \\ \sqrt{2}M_{12} \end{pmatrix} \quad \text{and} \quad \left[\underset{\sim}{\chi}\right] = \begin{pmatrix} \chi_{11} \\ \chi_{22} \\ \sqrt{2}\chi_{12} \end{pmatrix}$$

and the fourth-order compliance tensor $\underset{\approx}{\boldsymbol{d}}$ is reallocated in a matrix form

$$\left[\underset{\approx}{\boldsymbol{d}}\right] = \begin{pmatrix} d_{1111} & d_{2211} & \sqrt{2}d_{1211} \\ d_{2211} & d_{2222} & \sqrt{2}d_{1222} \\ \sqrt{2}d_{1211} & \sqrt{2}d_{1222} & 2d_{1212} \end{pmatrix}$$

so that the constitutive equation

$$\underset{\sim}{\chi} = \underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}} \quad \text{becomes} \quad \left[\underset{\sim}{\chi}\right] = \left[\underset{\approx}{\boldsymbol{d}}\right] \cdot \left[\underset{\sim}{\boldsymbol{M}}\right]$$

The same for $\underset{\approx}{\boldsymbol{D}}$ and $\underset{\approx}{\boldsymbol{C}}^{\sigma}$.

Voigt Notations

The constitutive sixth-order tensor $\underset{\approx}{\boldsymbol{h}}$ is turned into the 6×6 matrix $\left[\underset{\approx}{\boldsymbol{h}}\right]$:

$$\begin{pmatrix} h_{111111} & h_{111122} & \sqrt{2}h_{111121} & h_{111211} & h_{111222} & \sqrt{2}h_{111221} \\ h_{221111} & h_{221122} & \sqrt{2}h_{221121} & h_{221211} & h_{221222} & \sqrt{2}h_{221221} \\ \sqrt{2}h_{121111} & \sqrt{2}h_{121122} & 2h_{121121} & \sqrt{2}h_{121211} & \sqrt{2}h_{121222} & 2h_{121221} \\ h_{112111} & h_{112122} & \sqrt{2}h_{112121} & h_{112211} & h_{112222} & \sqrt{2}h_{112221} \\ h_{222111} & h_{222122} & \sqrt{2}h_{222121} & h_{222211} & h_{222222} & \sqrt{2}h_{222221} \\ \sqrt{2}h_{122111} & \sqrt{2}h_{122122} & 2h_{122121} & \sqrt{2}h_{122211} & \sqrt{2}h_{122222} & 2h_{122221} \end{pmatrix}$$

The third-order tensors $\underset{\frown}{\boldsymbol{\Gamma}}$ and $\underset{\frown}{\boldsymbol{R}}$ are reallocated in a vector form:

$$\left[\underset{\frown}{\boldsymbol{\Gamma}}\right] = \begin{pmatrix} \Gamma_{111} \\ \Gamma_{221} \\ \sqrt{2}\Gamma_{121} \\ \Gamma_{112} \\ \Gamma_{222} \\ \sqrt{2}\Gamma_{122} \end{pmatrix}, \quad \left[\underset{\frown}{\boldsymbol{R}}\right] = \begin{pmatrix} R_{111} \\ R_{221} \\ \sqrt{2}R_{121} \\ R_{112} \\ R_{222} \\ \sqrt{2}R_{122} \end{pmatrix} \quad \text{and} \quad \left[\underset{\frown}{\boldsymbol{\Gamma}}\right] = \left[\underset{\approx}{\boldsymbol{h}}\right] \cdot \left[\underset{\frown}{\boldsymbol{R}}\right]$$

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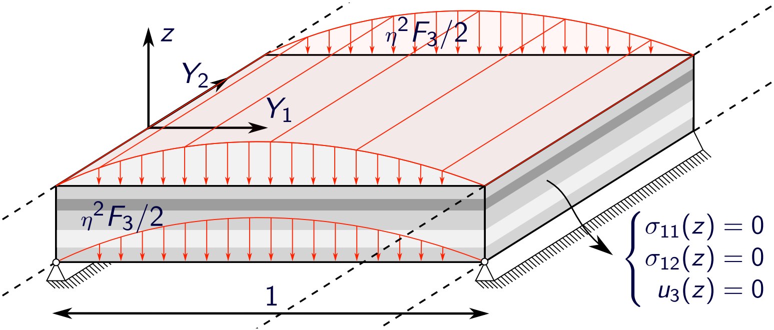
Pagano's boundary value problem (Pagano, 1969)

CFRP layers with different orientiations: $F_3(Y_1) = -F_0 \sin \kappa Y_1$ where $\lambda = 1/\kappa = \frac{1}{n\pi}$, $n \in \mathbb{N}^{+*}$ is the non-dimensional wavelength of the loading.



Invariant in x_2-Direction, "periodic" in x_1-Direction
$\Rightarrow$ No boundary layer!

Resolution of the Bending-Gradient problem

All the non-dimensional fields are invariant in Y_2-Direction
From $\chi_{\alpha\beta} = \Phi_{\alpha\beta\gamma,\gamma}$, we obtain:

$$\left[\underset{\sim}{\chi}\right] = \begin{pmatrix} \chi_{11} \\ \chi_{22} \\ \sqrt{2}\chi_{12} \end{pmatrix} = \begin{pmatrix} \Phi_{111,1} \\ \Phi_{221,1} \\ \sqrt{2}\Phi_{121,1} \end{pmatrix} = \begin{pmatrix} \Phi_{1,1} \\ \Phi_{2,1} \\ \Phi_{3,1} \end{pmatrix}$$

The equilibrium equations write as:

$$\left[\underset{\frown}{\boldsymbol{R}}\right] = \begin{pmatrix} R_{111} \\ R_{221} \\ \sqrt{2}R_{121} \\ R_{112} \\ R_{222} \\ \sqrt{2}R_{122} \end{pmatrix} = \begin{pmatrix} M_{11,1} \\ M_{22,1} \\ \sqrt{2}M_{12,1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad M_{11,11} = -F_3(Y_1)$$

Shear constitutive equation

Taking into account $R_{112} = R_{222} = R_{122} = 0$, $U_{3,2} = 0$, shear constitutive equation is rewritten in two parts.
A first part with unknowns involving active boundary conditions:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \eta^2 \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix} \cdot \begin{pmatrix} M_{11,1} \\ M_{22,1} \\ \sqrt{2} M_{12,1} \end{pmatrix} - \begin{pmatrix} U_{3,1} \\ 0 \\ 0 \end{pmatrix}$$

and a second part which enables the derivation of $\Phi_4 = \Phi_{112}$, $\Phi_5 = \Phi_{222}$, $\Phi_6 = \sqrt{2}\Phi_{122}$ on which no boundary condition applies:

$$\begin{pmatrix} \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{pmatrix} = \eta^2 \begin{pmatrix} h_{41} & h_{42} & h_{43} \\ h_{51} & h_{52} & h_{53} \\ h_{61} & h_{62} & h_{63} \end{pmatrix} \cdot \begin{pmatrix} M_{11,1} \\ M_{22,1} \\ \sqrt{2} M_{12,1} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ U_{3,1}/\sqrt{2} \end{pmatrix}$$

Final System

Finally, combining the above equations leads to the following set of equations which fully determines the problem:

$$\begin{cases} M_{11,11} = F_0 \sin \kappa Y_1 \\ \left[\underset{\approx}{\boldsymbol{d}}\right] \cdot \left[\underset{\sim}{\boldsymbol{M}}\right] - \eta^2 \underset{\sim}{\boldsymbol{h}} \cdot \left[\underset{\sim}{\boldsymbol{M}}\right]_{,11} = \begin{pmatrix} U_{3,11} \\ 0 \\ 0 \end{pmatrix} \\ \left[\underset{\sim}{\boldsymbol{M}}\right] = 0 \quad \text{for} \quad Y_1 = 0 \quad \text{and} \quad Y_1 = 1 \\ U_3 = 0 \quad \text{for} \quad Y_1 = 0 \quad \text{and} \quad Y_1 = 1 \end{cases}$$

where for convenience, $\underset{\sim}{\boldsymbol{h}}$ is the 3×3 submatrix of $\left[\underset{\frown}{\boldsymbol{h}}\right]$:

$$\underset{\sim}{\boldsymbol{h}} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix}$$

Solution

This differential system is well-posed and the solution is unique. Its is of the form:

$$\left[\underset{\sim}{\boldsymbol{M}}\right] = \left[\underset{\sim}{\boldsymbol{M}}^*\right] \sin \kappa Y_1 \quad \text{and} \quad U_3 = U_3^* \sin \kappa Y_1$$

where $\left[\underset{\sim}{\boldsymbol{M}}^*\right]$ and U_3^* are constants explicitly known in terms of the problem inputs.

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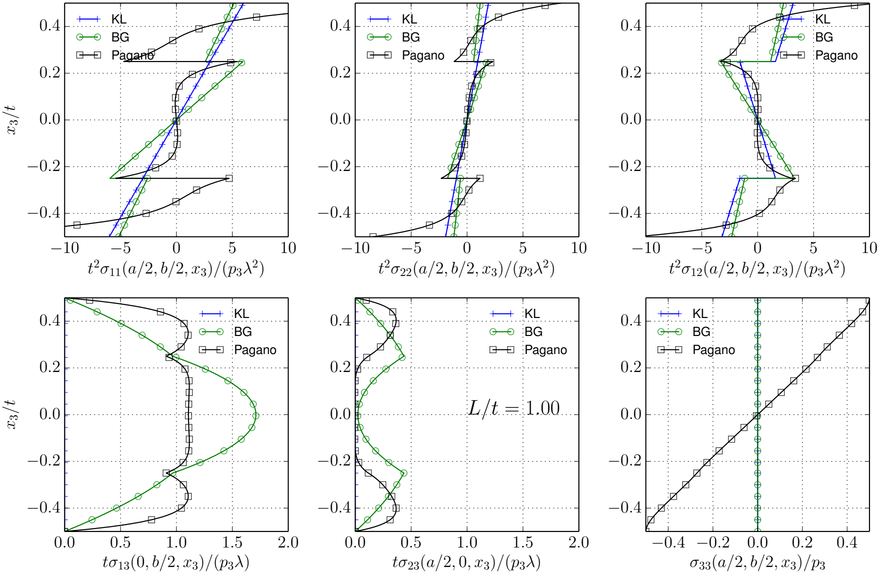
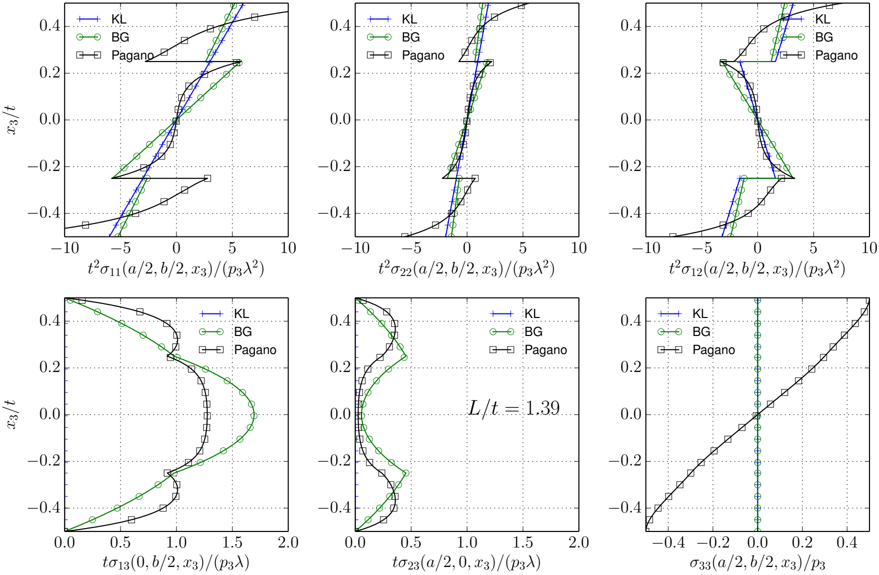
Stress distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



Stress distributions for a $[30°, -30°, 30°]$ stack



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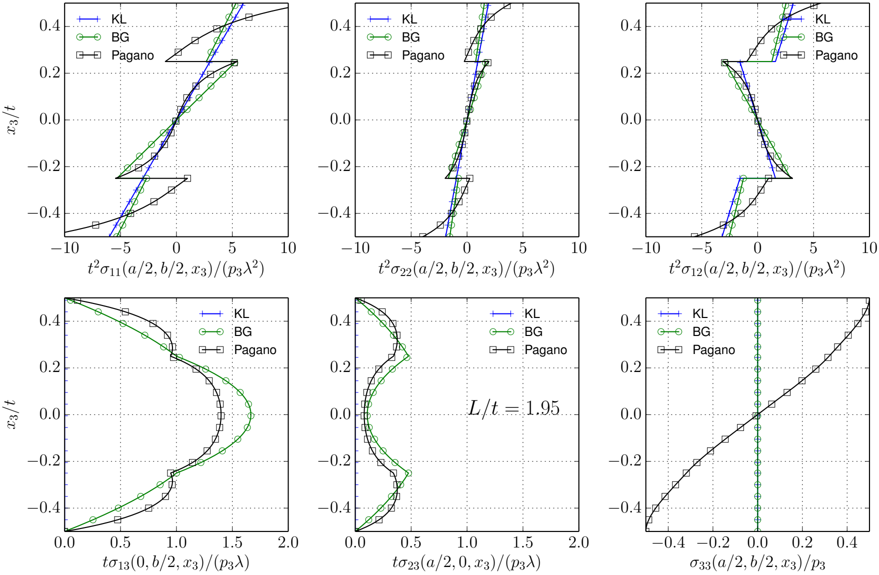

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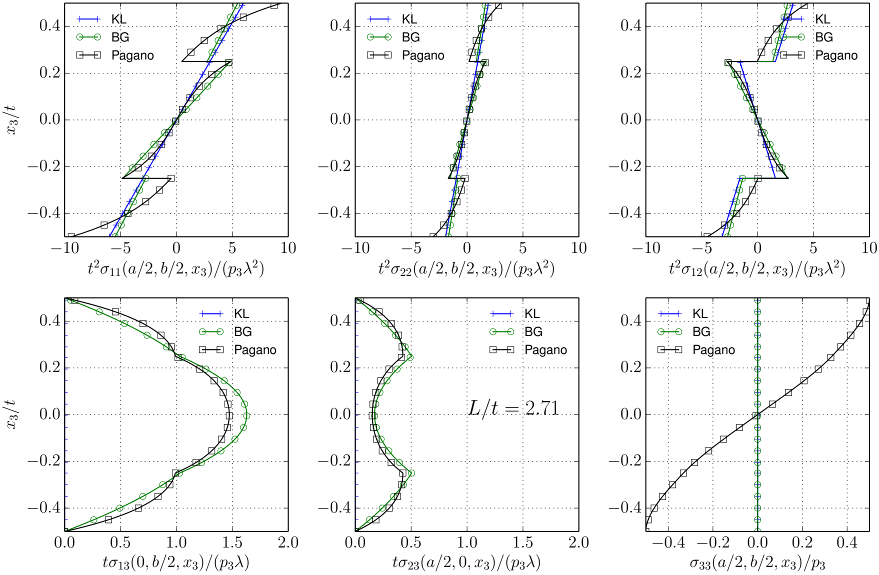
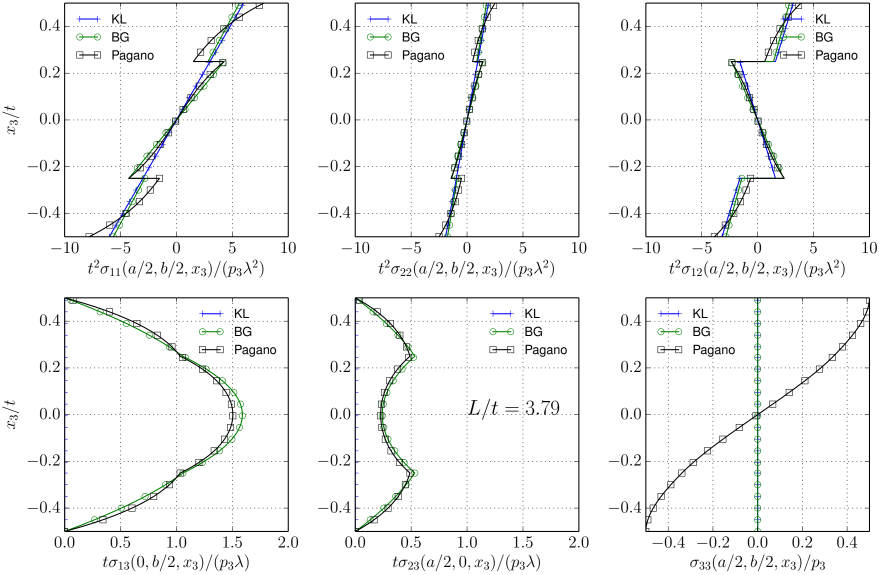


Stress distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



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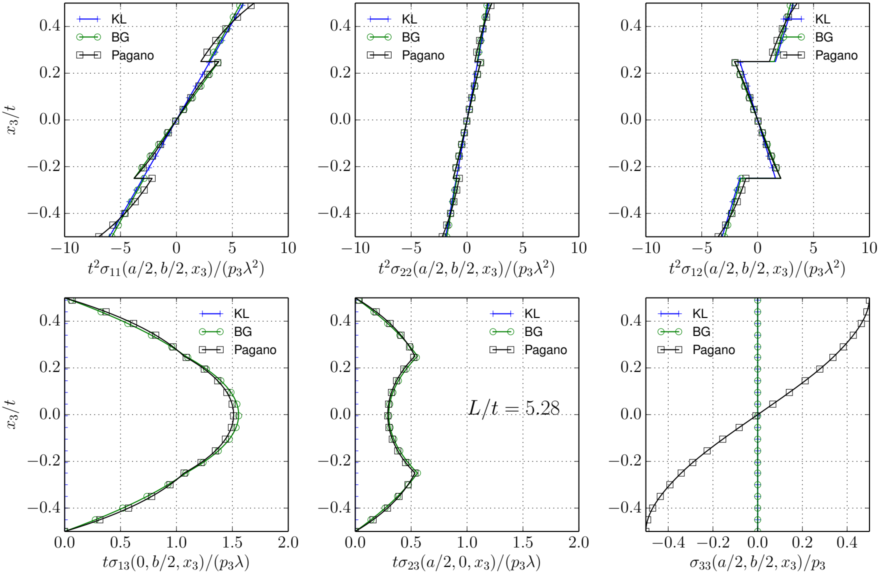

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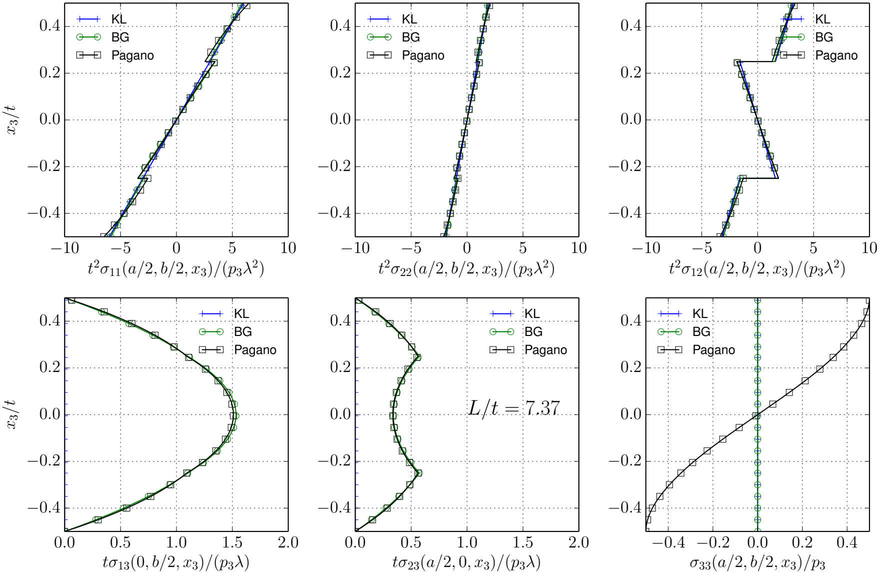

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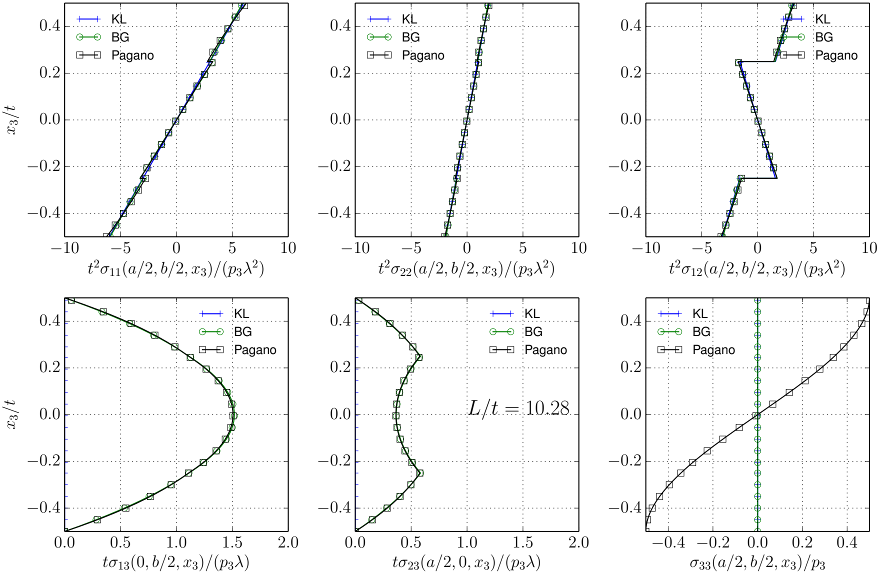

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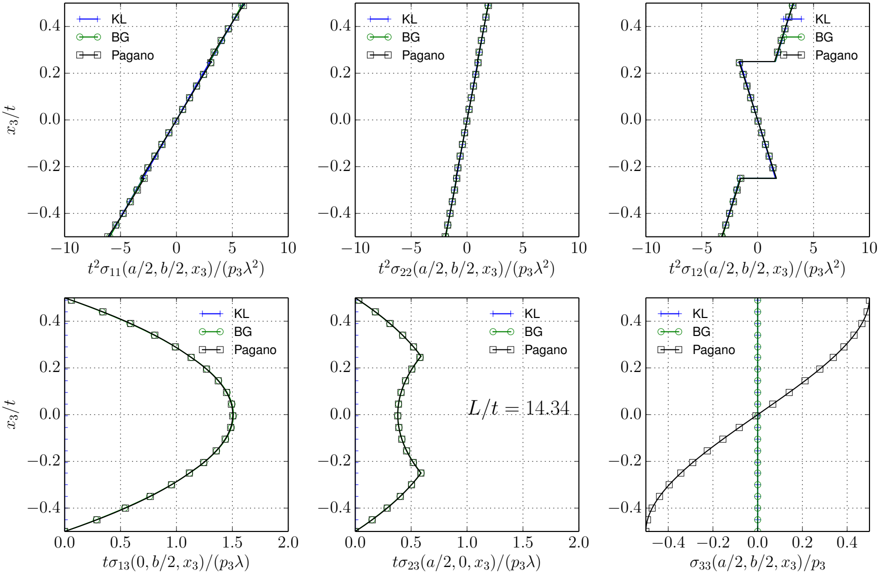
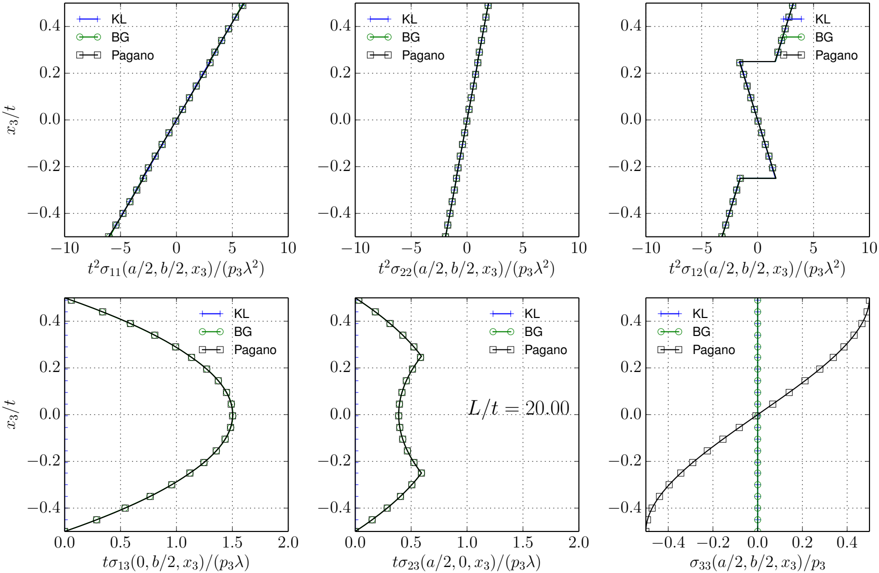
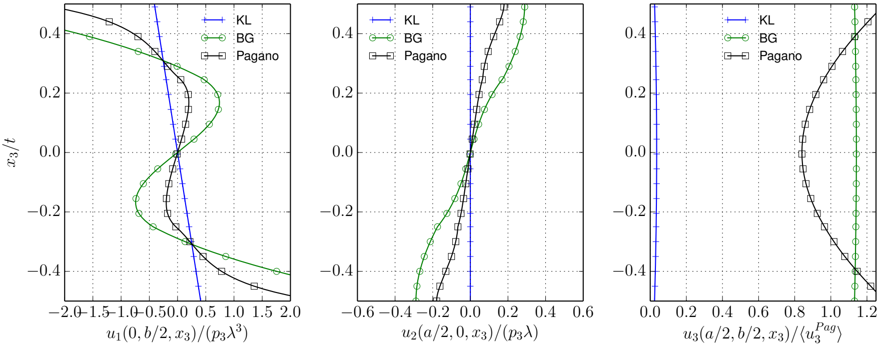


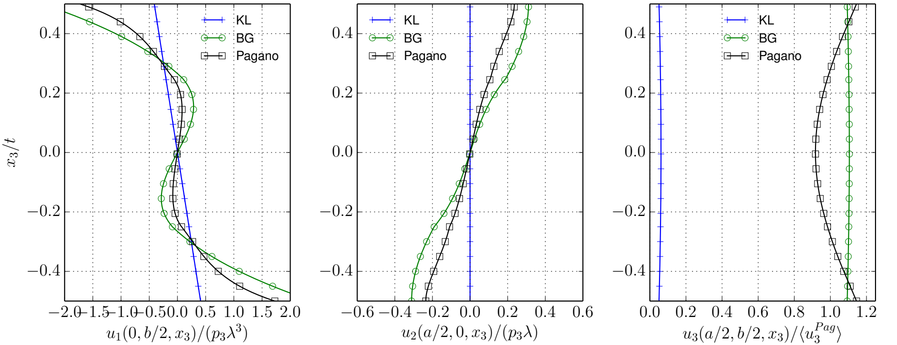
Stress distributions for a $[30°, -30°, 30°]$ stack



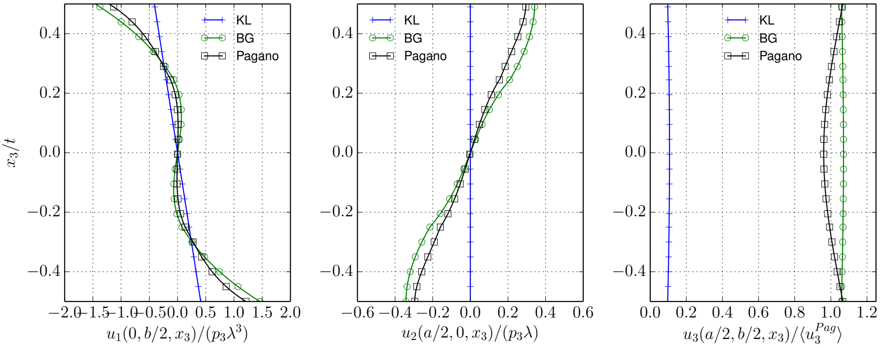
Displacement distributions for a $[30°, -30°, 30°]$ stack



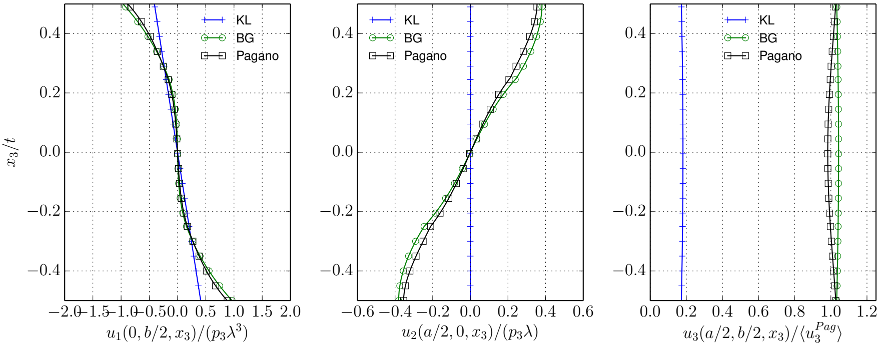
Displacement distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



Displacement distributions for a $[30°, -30°, 30°]$ stack



Displacement distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



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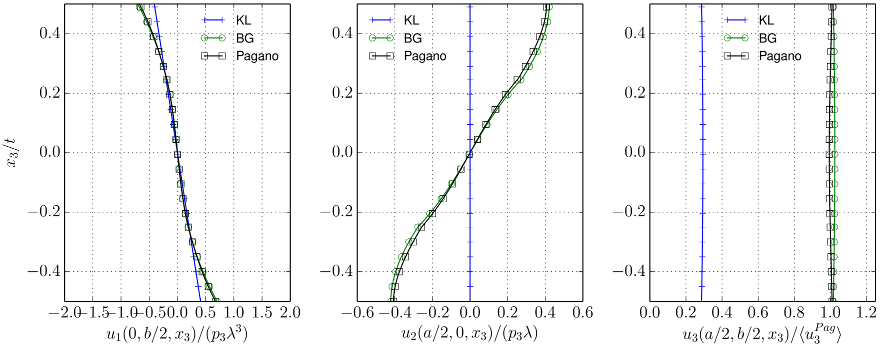
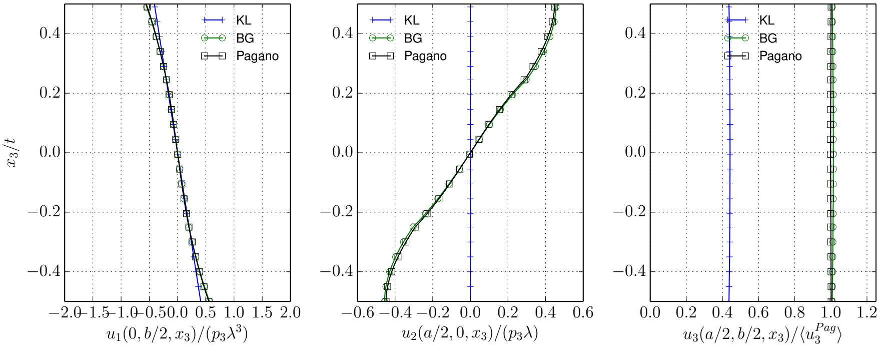


Displacement distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



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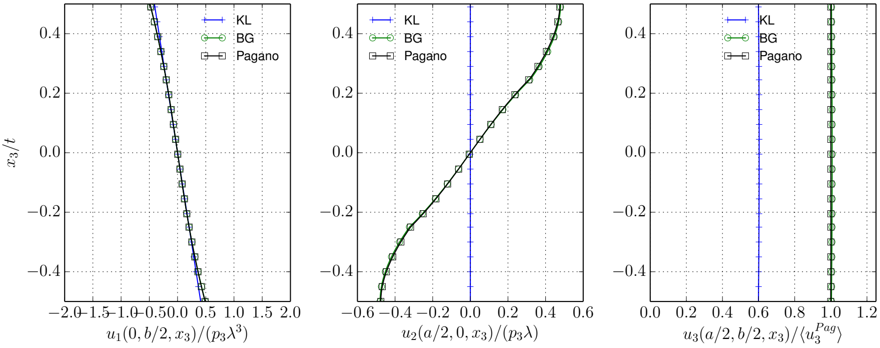

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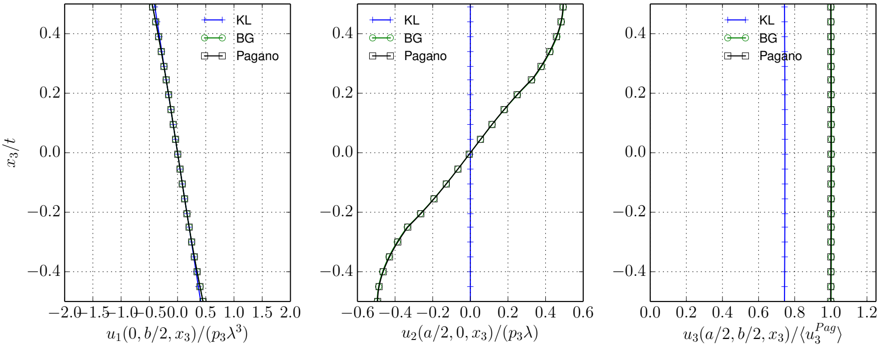

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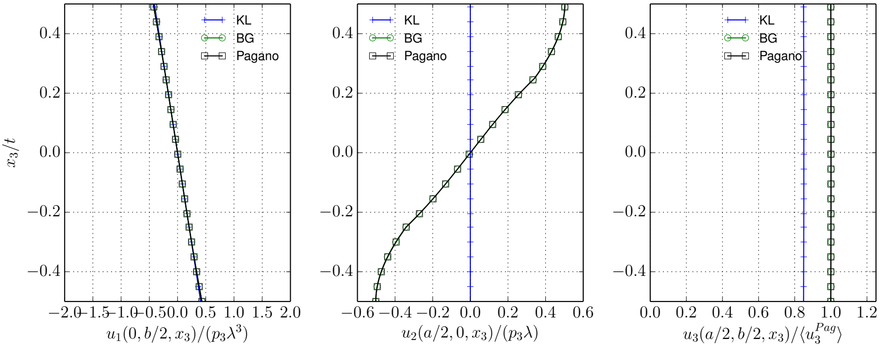

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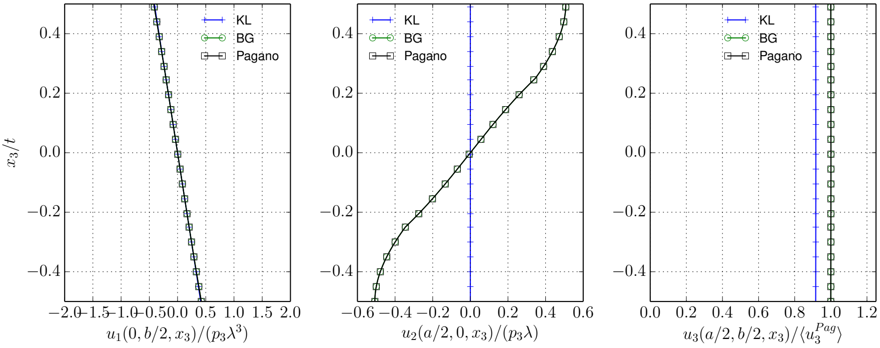
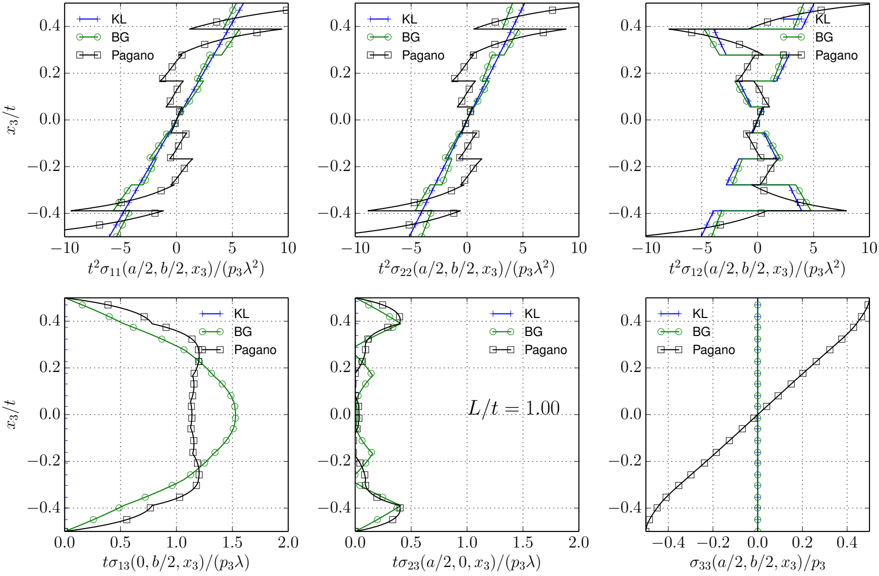
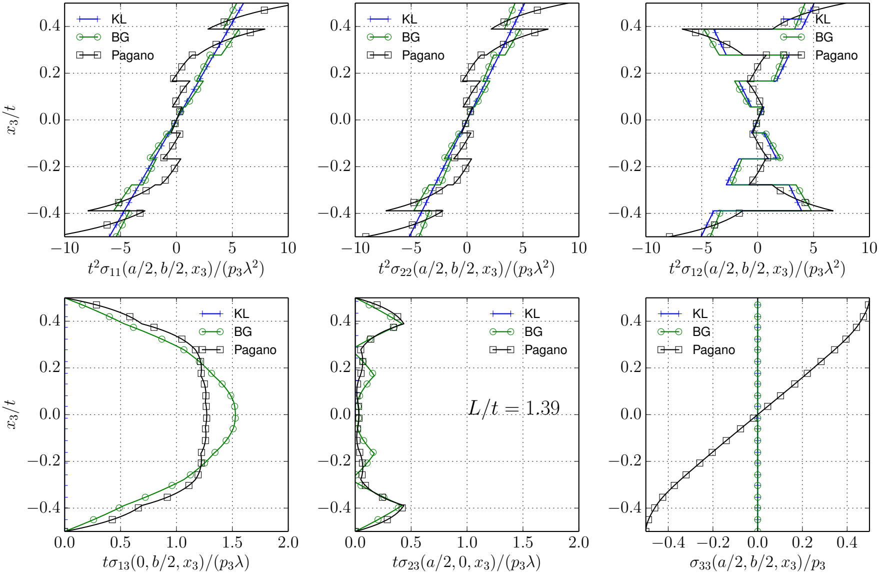


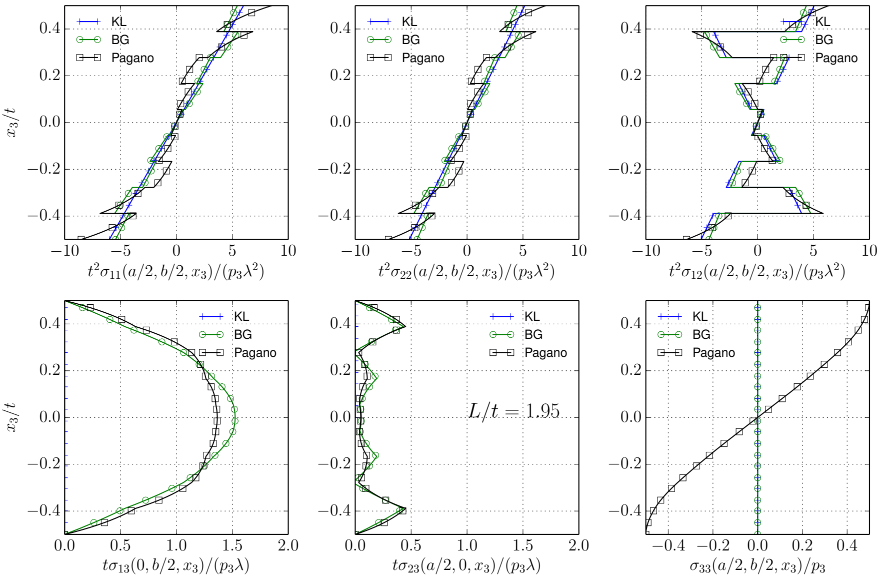
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



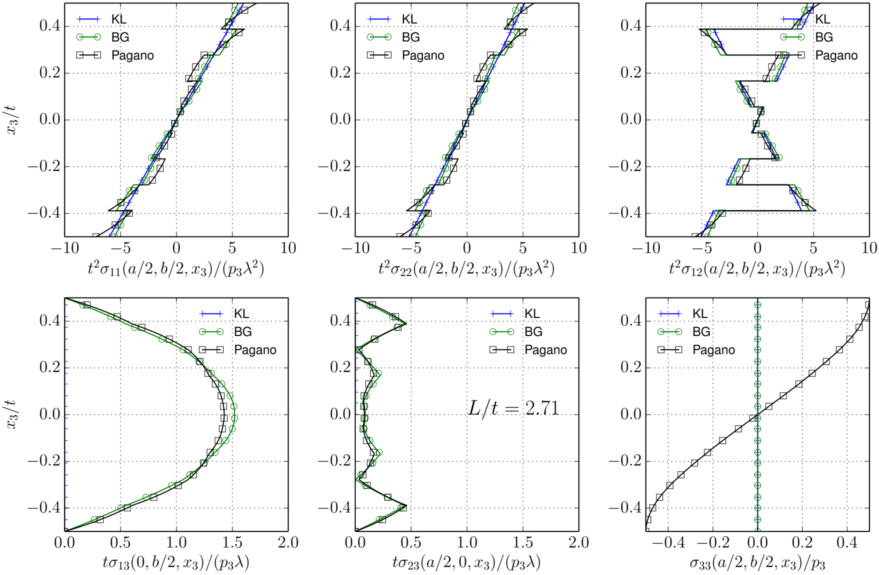
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



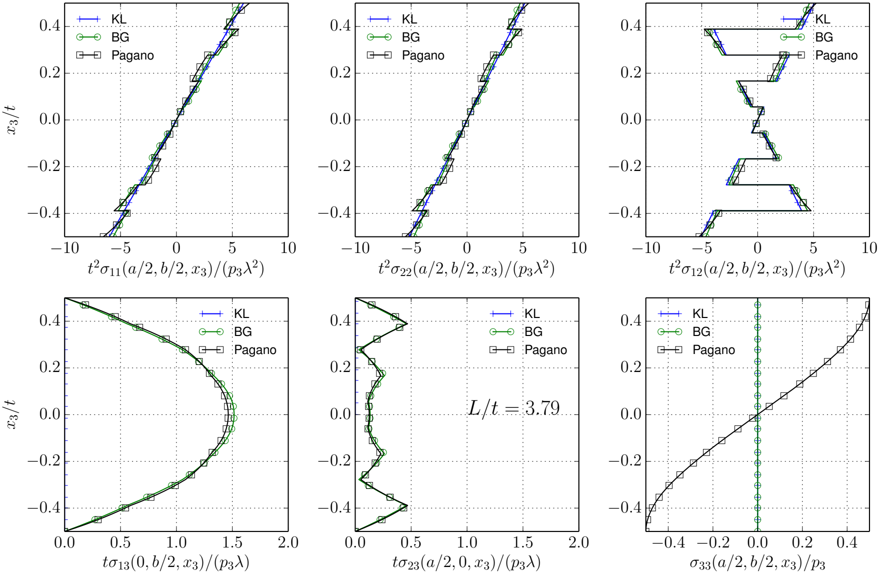
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



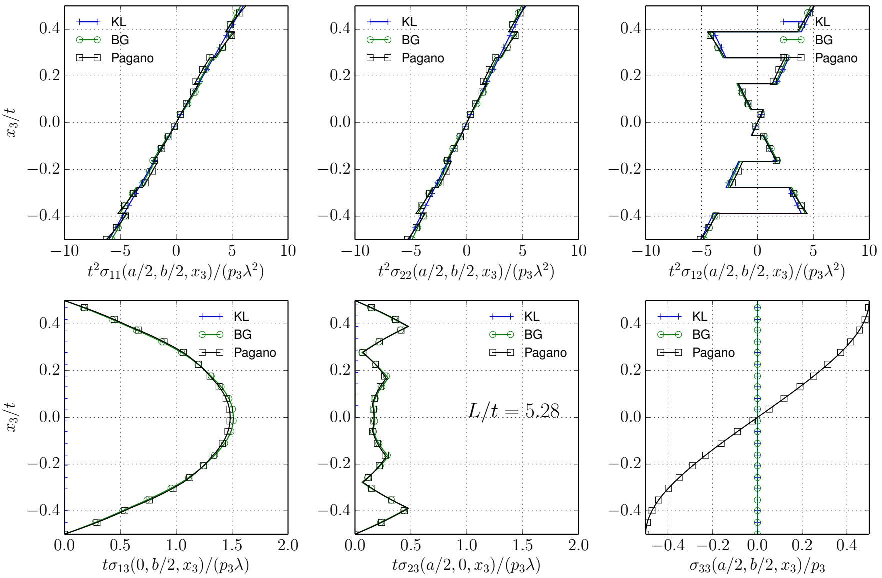
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



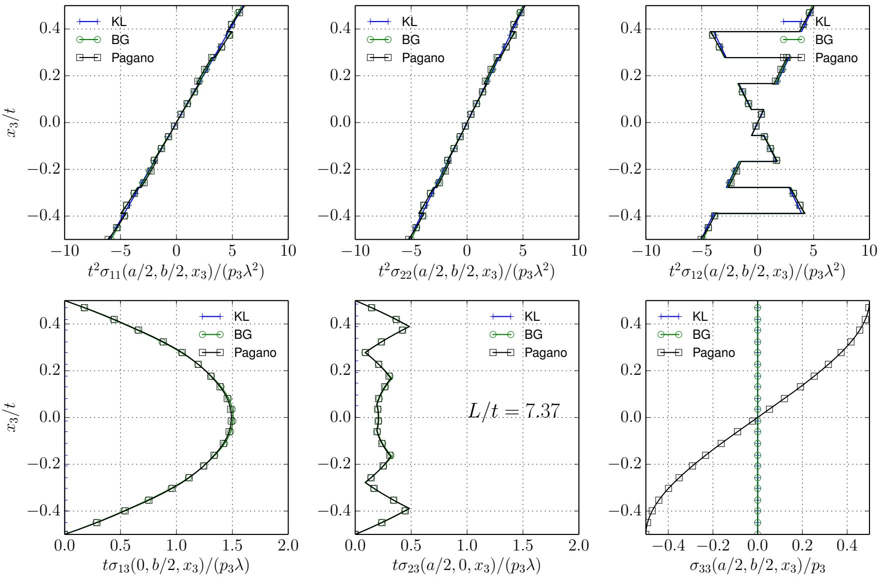
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



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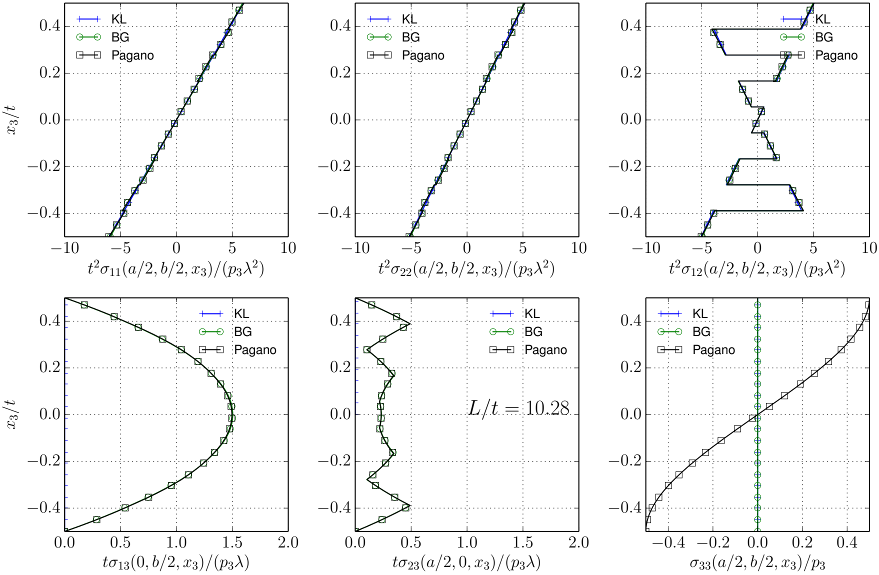
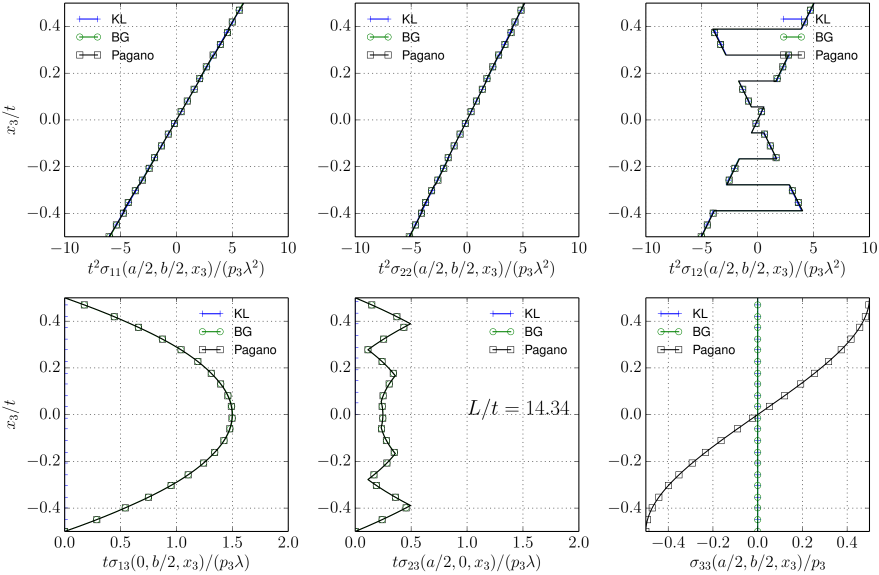
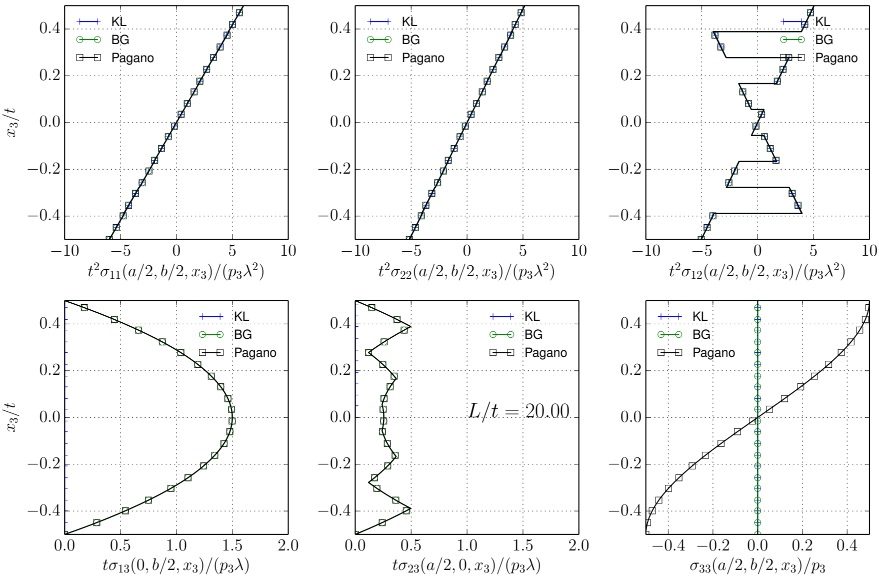


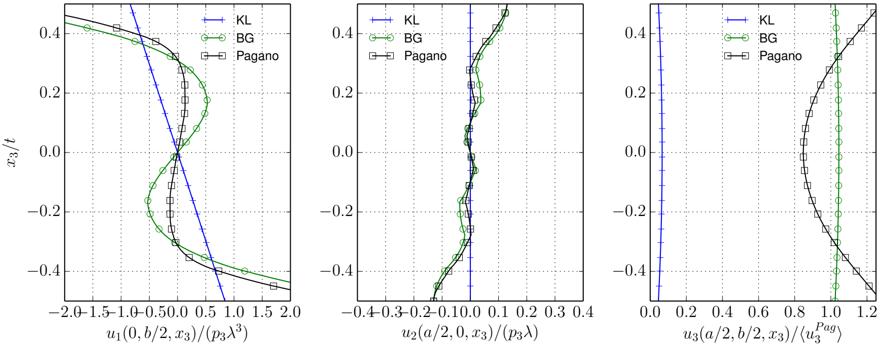
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



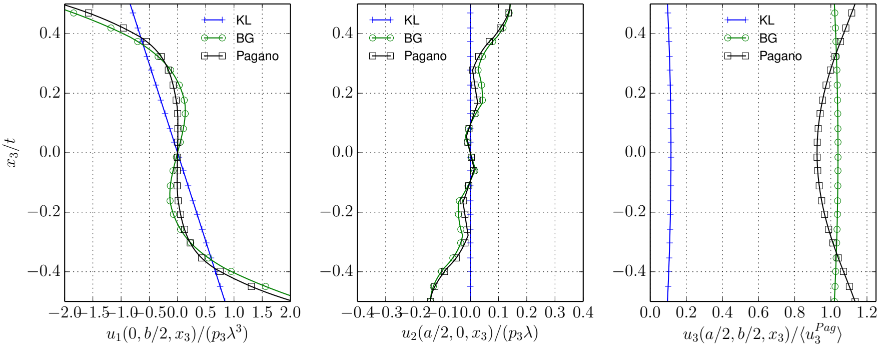
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Displacement distributions for a $[45°, -45°]^4, 45°$ stack



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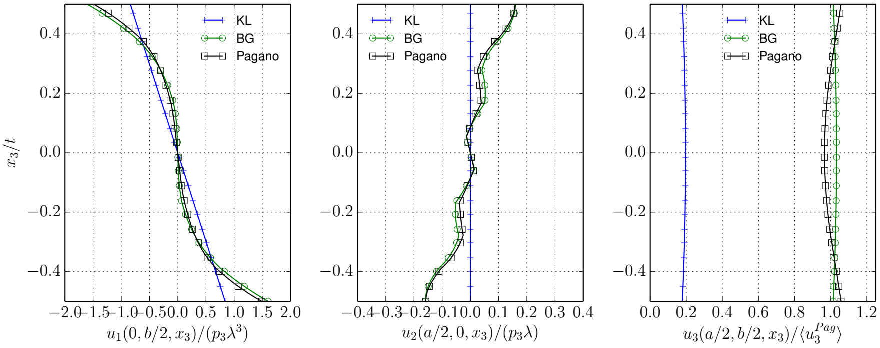
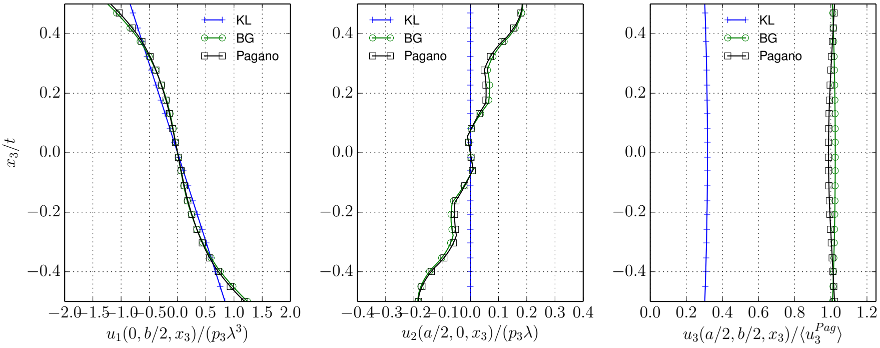
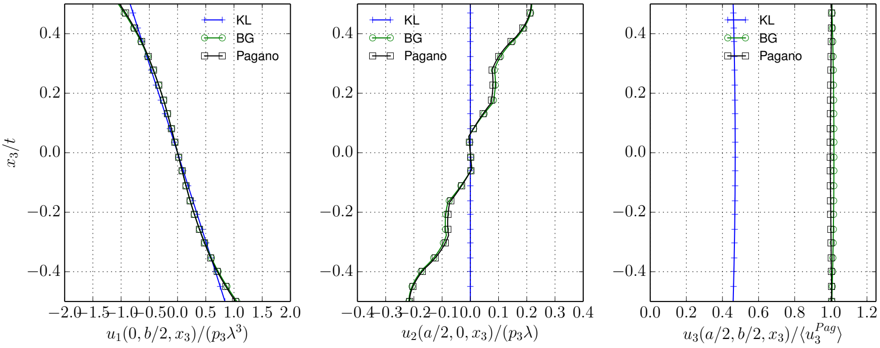


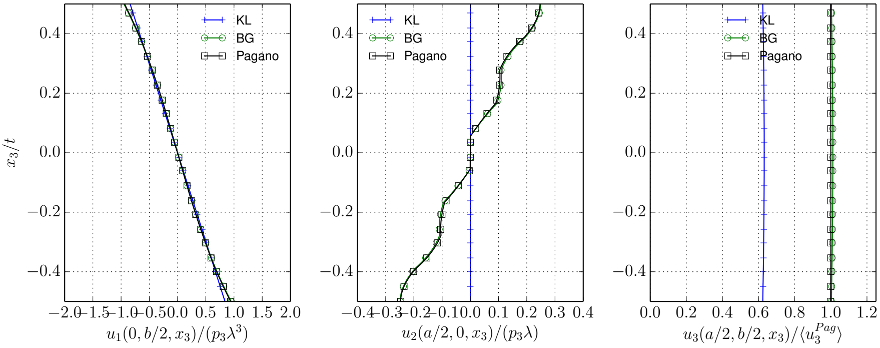
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



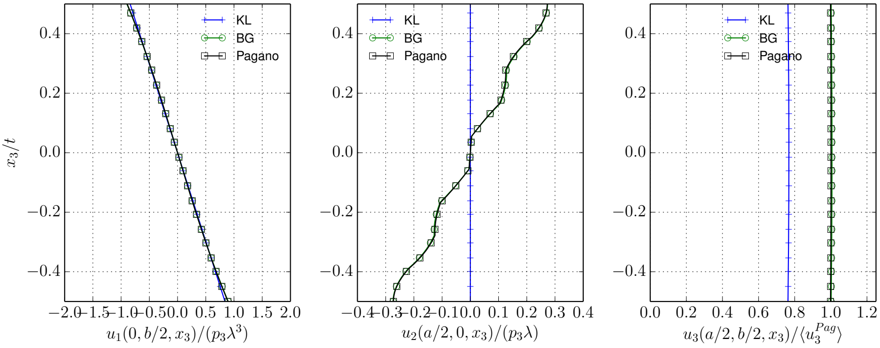
Displacement distributions for a $[45°, -45°]^4, 45°$ stack



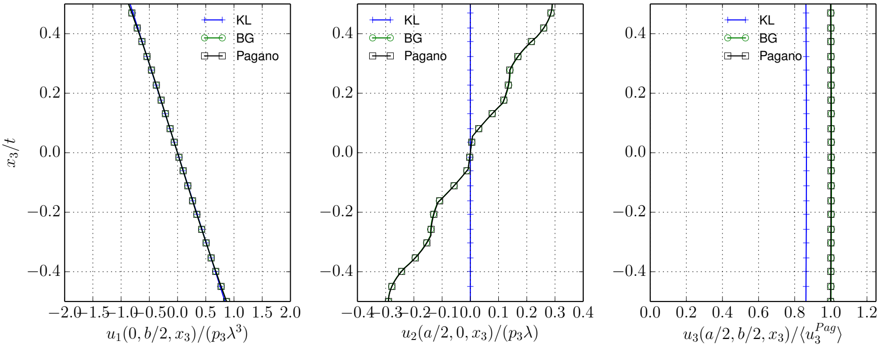
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



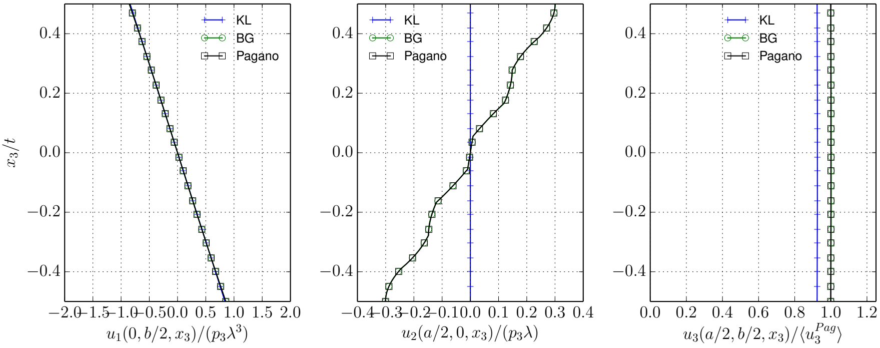
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



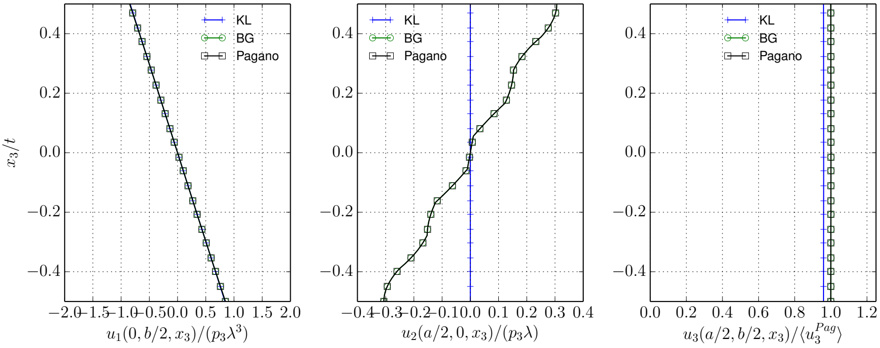
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



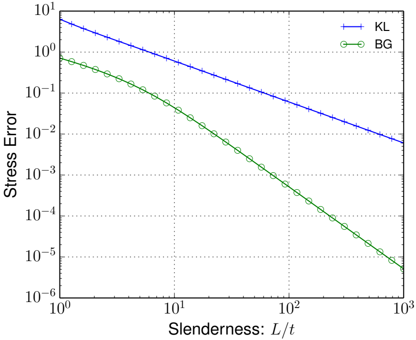
Displacement distributions for a $[45°, -45°]^4, 45°$ stack

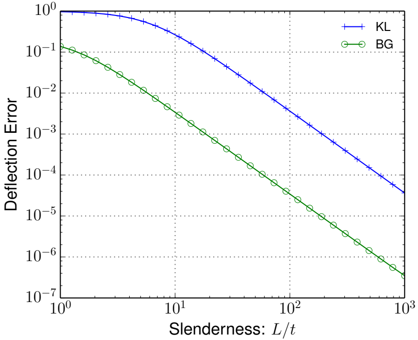


Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Convergence for a $[30°, -30°, 30°]$ stack





$\Delta(\sigma)$ rate: KL $\sim t$ and BG $\sim t^2$

$\Delta(U_3)$ rate: KL $\sim t^2$ and BG $\sim t^2$

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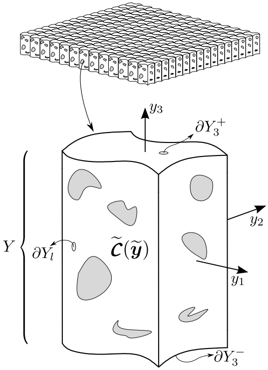
Assumptions

- The same 3D problem as for laminates but $\underset{\approx}{\boldsymbol{C}}^{t}(\underline{\boldsymbol{x}})$ depends now on the three coordinates.
- $\underset{\approx}{\boldsymbol{C}}^{t}(\underline{\boldsymbol{x}})$ is periodic in the two first coordinates (x_1, x_2).
- The in-plane dimension of the unit cell is comparable to its thickness t.
- t is small with respect to the in-plane dimension of the plate L.

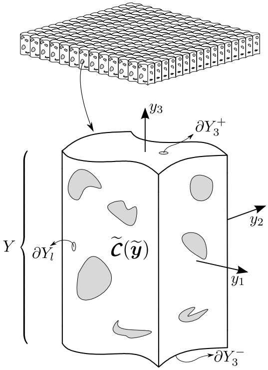


$$
\begin{cases}
\underset{\sim}{\boldsymbol{\sigma}}^{t} \cdot \underline{\boldsymbol{\nabla}} = 0 & \text{on } \Omega^{t}. \\
\underset{\sim}{\boldsymbol{\sigma}}^{t}(\underline{\boldsymbol{x}}) = \underset{\approx}{\boldsymbol{C}}^{t}(\underline{\boldsymbol{x}}) : \underset{\sim}{\boldsymbol{\varepsilon}}^{t}(\underline{\boldsymbol{x}}) & \text{on } \Omega^{t}. \\
\underset{\sim}{\boldsymbol{\sigma}}^{t} \cdot \left(\pm \underline{\boldsymbol{e}}_3\right) = f_3 \underline{\boldsymbol{e}}_3 & \text{on } \omega^{L\pm}. \\
\underset{\sim}{\boldsymbol{\varepsilon}}^{t} = \underline{\boldsymbol{u}}^{t} \otimes^{s} \underline{\boldsymbol{\nabla}} & \text{on } \Omega^{t}. \\
\underline{\boldsymbol{u}}^{t} = 0 & \text{on } \partial\omega^{L} \times]-t/2, t/2[
\end{cases}
$$

Extension to periodic plates

- Bending auxiliary problem (Caillerie, 1984)

$$\mathcal{P}^K \begin{cases} \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{\nabla}} = 0 \\ \underset{\sim}{\boldsymbol{\sigma}}^K = \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{y}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}^K \\ \underset{\sim}{\boldsymbol{\varepsilon}}^K = y_3 \underset{\sim}{\boldsymbol{K}} + \underline{\boldsymbol{\nabla}} \otimes^s \underline{\boldsymbol{u}}^{\text{per}} \\ \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{e}}_3 = 0 \text{ on free faces } \partial Y_3^{\pm} \\ \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{n}} \text{ skew-periodic on lateral edge } \partial Y_l \\ \underline{\boldsymbol{u}}^{\text{per}}(\underline{\boldsymbol{y}}) \; (y_1, y_2)\text{-periodic on lateral edge } \partial Y_l \end{cases}$$



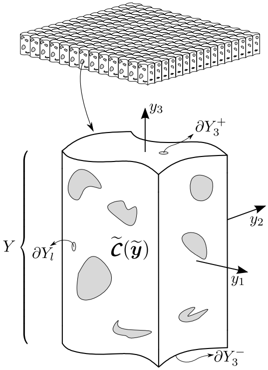
$\rightarrow$ gives:

Localization $\underline{\boldsymbol{u}}^K$ $\underset{\sim}{\boldsymbol{\sigma}}^K$ related to the curvature $\underset{\sim}{\boldsymbol{K}}$
Bending stiffness: $\underset{\approx}{\boldsymbol{D}}$

Extension to periodic plates

- Bending auxiliary problem (Caillerie, 1984)
- Shear auxiliary problem

$$\mathcal{P}^R \begin{cases} \underset{\sim}{\sigma}^R \cdot \underline{\nabla}_y + \underset{\sim}{\sigma}^M \cdot \underline{\nabla}_Y = 0 \\ \underset{\sim}{\sigma}^R = \underset{\approx}{C}\left(\underline{y}\right) : \left(\underline{u}^M \otimes^s \underline{\nabla}_Y + \underline{u}^R \otimes^s \underline{\nabla}_y\right) \\ \underset{\sim}{\sigma}^R \cdot \underline{e}_3 = 0 \text{ on free faces } \partial Y_3^{\pm} \\ \underset{\sim}{\sigma}^R \cdot \underline{n} \text{ skew-periodic on lateral edge } \partial Y_l \\ \underline{u}^R(\underline{y}) \ (y_1, y_2)\text{-periodic on lateral edge } \partial Y_l \end{cases}$$



$\rightarrow$ gives:

Localization $\underline{u}^R$ and $\underset{\sim}{\sigma}^R$ related to $\underset{\frown}{R}$
Shear compliance tensor: $\underset{\frown\frown}{h}$

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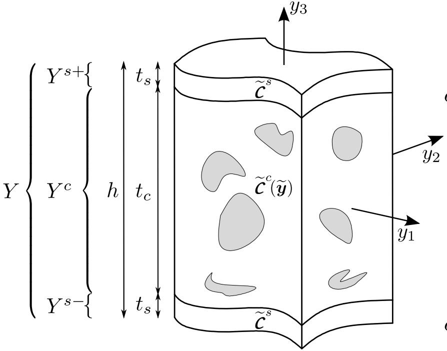
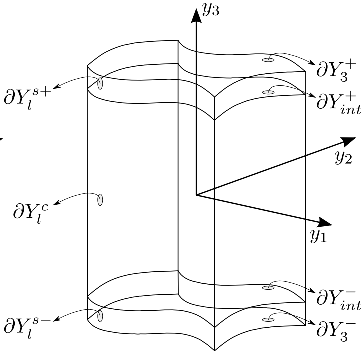
Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)



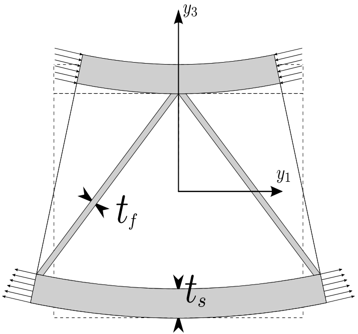


Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)
- Bending auxiliary problem
  - Contrast assumption $\Leftrightarrow t_f \ll t_s$:
    $\rightarrow t_s/t_f$ Contrast ratio

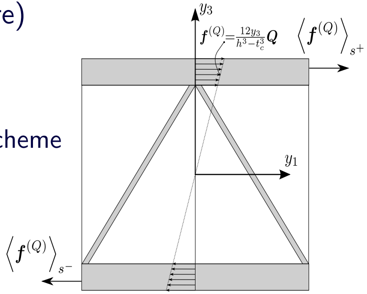
$\Rightarrow$ Skins under traction/compression

$\Rightarrow$ Core not involved in Bending stiffness



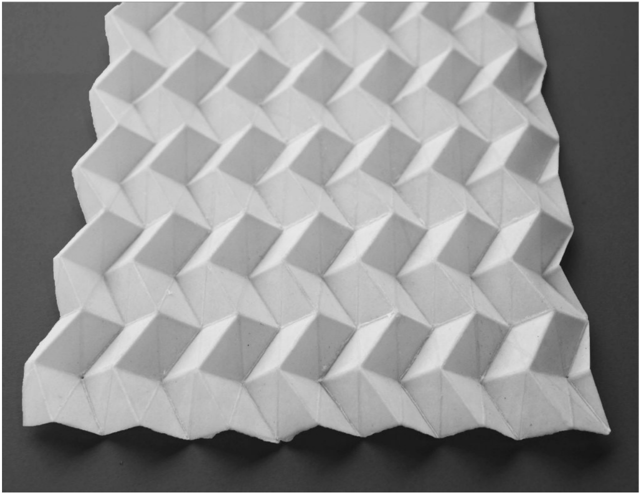
Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)
- Bending auxiliary problem
- Shear auxiliary problem
  - $\underline{\boldsymbol{f}}^R$ becomes $\underline{\boldsymbol{f}}^{(Q)}$ + Direct homogenization scheme
  - The BG is degenerated into RM model
  - $\underline{\boldsymbol{f}}^{(Q)}$ confirms the classical intuition



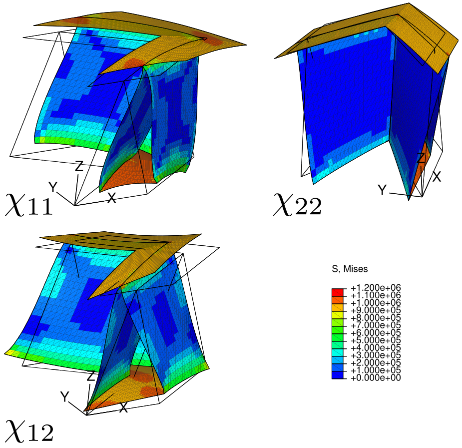
Lebée and Sab (2012a)

Application to the chevron pattern



Application to the chevron pattern

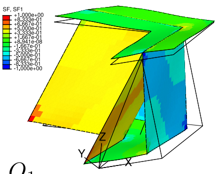
Bending:



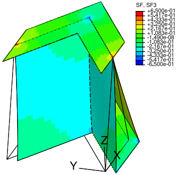
Application to the chevron pattern

Shear forces localization $\underset{\sim}{\boldsymbol{\sigma}}^{(Q)}$

- Overall shearing of the core



Q_1

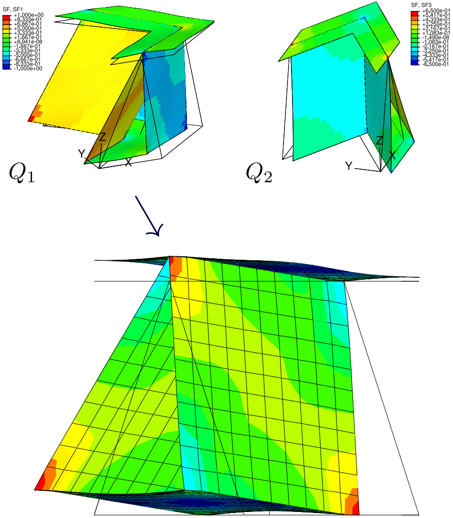


Q_2

Application to the chevron pattern

Shear forces localization $\underset{\sim}{\boldsymbol{\sigma}}^{(Q)}$

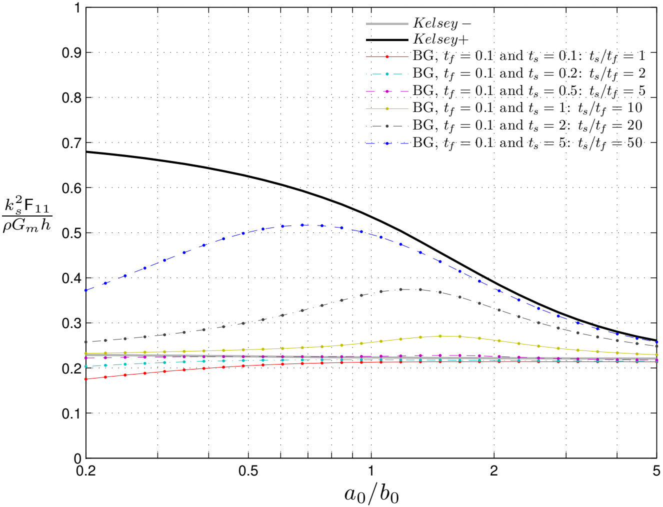
- Overall shearing of the core
- Out-of-plane skins distortion



Application to the chevron pattern

Shear forces localization $\underset{\sim}{\sigma}^{(Q)}$

- Overall shearing of the core
- Out-of-plane skins distorsion
- Critically influence shear force stiffness



Lebée and Sab (2012b)

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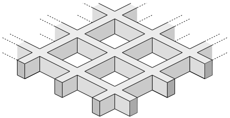
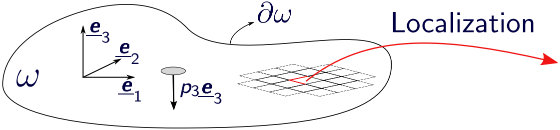
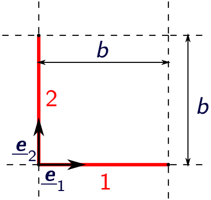
Homogenizing an orthogonal beam lattice



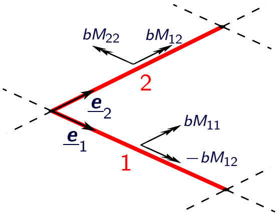
$=$

Thick-plate model (macro) $+$ 2 St-Venant Beams (micro)





Field localization



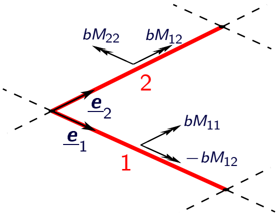
Bending moment $\left(\underline{r}^{(M)}, \underline{m}^{(M)}\right)$:

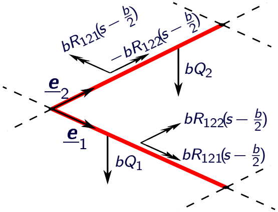
Apply $\underset{\sim}{M}$ "on average" on the unit-cell (Caillerie, 1984)

$$^1\underline{r}^{(M)} = {}^2\underline{r}^{(M)} = \underline{0}$$

$$^1\underline{m}^{(M)} = \begin{pmatrix} -bM_{12} \\ bM_{11} \\ 0 \end{pmatrix}_1 \quad \text{and} \quad ^2\underline{m}^{(M)} = \begin{pmatrix} bM_{12} \\ bM_{22} \\ 0 \end{pmatrix}_2$$

Field localization





Bending moment $\left(\underline{\boldsymbol{r}}^{(M)}, \underline{\boldsymbol{m}}^{(M)}\right)$:

Apply $\underset{\sim}{\boldsymbol{M}}$ "on average" on the unit-cell (Caillerie, 1984)

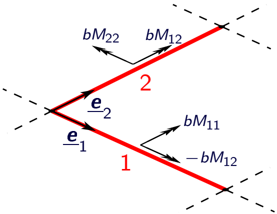
Bending gradient $\left(\underline{\boldsymbol{r}}^{(R)}, \underline{\boldsymbol{m}}^{(R)}\right)$:

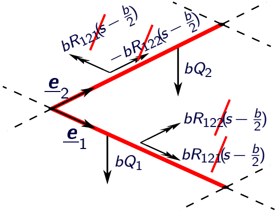
Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma} x_\gamma$ (Lebée and Sab, 2013a)

$$
{}^1\underline{\boldsymbol{r}}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b\underbrace{\left(R_{111} + R_{122}\right)}_{Q_1} \end{pmatrix}_1 \qquad {}^1\underline{\boldsymbol{m}}^{(R)} = \begin{pmatrix} bR_{121}\left(s - \frac{b}{2}\right) \\ bR_{122}\left(s - \frac{b}{2}\right) \\ 0 \end{pmatrix}_1
$$

$$
{}^2\underline{\boldsymbol{r}}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b\underbrace{\left(R_{121} + R_{222}\right)}_{Q_2} \end{pmatrix}_2 \qquad {}^2\underline{\boldsymbol{m}}^{(R)} = \begin{pmatrix} -bR_{122}\left(s - \frac{b}{2}\right) \\ bR_{121}\left(s - \frac{b}{2}\right) \\ 0 \end{pmatrix}_2
$$

Field localization





Bending moment $\left(\underline{\boldsymbol{r}}^{(M)}, \underline{\boldsymbol{m}}^{(M)}\right)$:

Apply $\underset{\sim}{\boldsymbol{M}}$ ”on average” on the unit-cell (Caillerie, 1984)

Bending gradient $\left(\underline{\boldsymbol{r}}^{(R)}, \underline{\boldsymbol{m}}^{(R)}\right)$:

Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma} x_\gamma$ (Lebée and Sab, 2013a)

Reissner-Mindlin $\left(\underline{\boldsymbol{r}}^{(Q)}, \underline{\boldsymbol{m}}^{(Q)}\right)$:

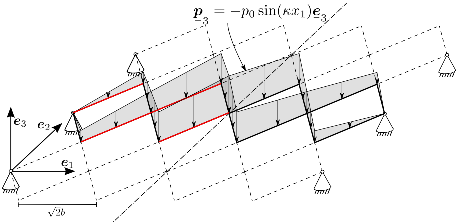
Assume cylindrical bending (Whitney, 1969; Cecchi and Sab, 2007)

$$Q_1 = R_{111},\ Q_2 = R_{222}, \quad R_{121} = R_{122} = R_{221} = R_{112} = 0$$

$${}^1\underline{\boldsymbol{r}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_1 \end{pmatrix}_1 \quad \text{and} \quad {}^1\underline{\boldsymbol{m}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_1$$

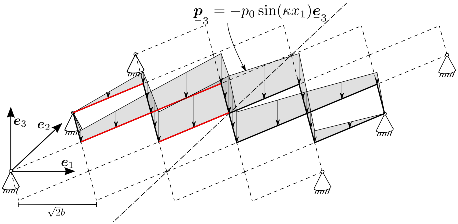
$${}^2\underline{\boldsymbol{r}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_2 \end{pmatrix}_2 \quad \text{and} \quad {}^2\underline{\boldsymbol{m}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_2$$

Application: lattice rotated 45° and cylindrical bending

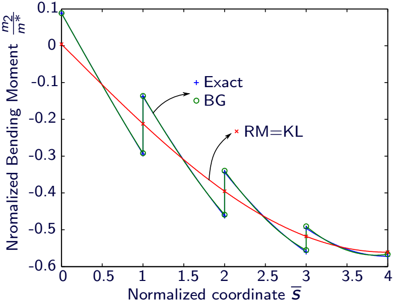


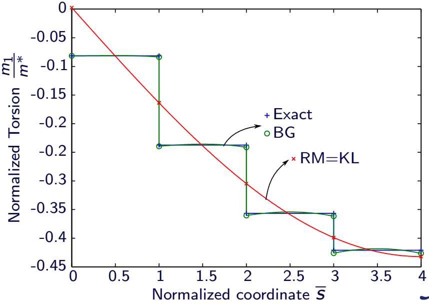
- Exact solution
- Plate solution + Localization (RM and BG)

Application: lattice rotated 45° and cylindrical bending



- Exact solution
- Plate solution + Localization (RM and BG)





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